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Cycles and harmonic forms on locally symmetric spaces .... John J. Millson 3
On the existence of restricted K-limits .... Urban Cegrell 39
Residual canonical correlations .... Anant M. Kshirsagar and R. P. Gupta 46
More on convergence of continuous functions and topological convergence of sets .... Gerald Beer 52
Two inequalities for planar convex sets .... George Tsintsifas 60
Extended centroids of skew polynomial rings..... Jerry D. Rosen and Mary Peles Rosen 67
On the Pontrjagin algebra of a certain class of flags of foliations .... F. J. Carreras and A. M. Naveira 77
Finite groups which are automorphism groups of infinite groups only .... Jay Zimmerman 84
Finite intersections of PID or factorial overrings .... D. D. Anderson and David F. Anderson 91
Integrally closed condensed domains are Bezout .... David F. Anderson, Jimmy T. Arnold and David E. Dobbs 98
Another class of cyclicly extensible and reducible properties .... B. Lehman 103
On quasisimilarity for Toeplitz operators .... K. Seddighi 107
The Dirichlet problem with Denjoy-Perron integrable boundary condition .... M. Benedicks and W. F. Pfeffer 113
Structure of certain periodic rings .... Hazar Abu-Khuzam and Adil Yaqub 120
The fibre of the double suspension is an H-space .... Paul Selick 124

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CYCLES AND HARMONIC FORMS ON LOCALLY SYMMETRIC SPACES

BY

JOHN J. MILLSON

ABSTRACT. Two constructions of cohomology classes for congruence subgroups of unit groups of quadratic forms over totally real number fields are given and shown to coincide. One is geometric, using cycles, and the other is analytic, using the oscillator (Weil) representation. Considerable background material on this representation is given.

0. **Introduction**. This paper is a considerably expanded version of the Coxeter-James Lecture which I delivered at the Canadian Mathematical Congress in Victoria, British Columbia on December 12, 1981. The main theme of the paper is the equality between the harmonic forms arising as the Poincaré duals of totally geodesic cycles for certain locally symmetric spaces and the harmonic forms on these same spaces constructed via the oscillator (or Weil) representation. This relationship is precisely stated (in the orthogonal case) in the Main Theorem — Chapter III, Section 2. I also tried to present foundational material concerning the oscillator representation and theta functions and particularly the splitting of the metaplectic extension in a more invariant way than usual.

There are three chapters in the paper. The first chapter presents the cycles — generalizations of a closed geodesic for the Poincaré metric on a Riemann surface of genus greater than 1.

The second chapter deals with the oscillator representation and the functional equation of the theta function. I prove that if Q is a non-degenerate quadratic form over $\mathbb{Z}/2$, Γ_Q is the group of integral symplectic matrices which when reduced modulo 2 give isometries of Q and $\tilde{\Gamma}_Q$ is the metaplectic extension then there is a theta multiplier $\epsilon:\tilde{\Gamma}_Q \to Q/\mathbb{Z}$. By this I mean that for each Q there exists a tempered distribution Θ on R^n so that for all $\gamma \in \tilde{\Gamma}_Q$: $\omega(\gamma)\Theta = \epsilon(\gamma)^{-1}\Theta$ where ω is the oscillator representation. Θ and ϵ depend on Q. In fact, Dennis Johnson and I have proved that the commutator quotient of $\tilde{\Gamma}_Q$ is $\mathbb{Z}/8$ (assuming that Q is a form in 6 or more variables) and consequently ϵ takes values in the 8th roots of unity. This result combined with the remarks at the end of Chapter II, Section 3, gives an explanation for the presence of 8th roots

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of unity in the transformation law for the classical theta function (considered as a function of the period matrix). Indeed, in case Q is the split form, ϵ is the usual theta multiplier. In this case I show that ϵ becomes trivial over the principal congruence subgroup of level 4, Theorem II.3.2, by using the usual Steinberg relations. In case Q is the anisotropic form in 2 variables, then $\Gamma_Q = SL_2(Z)$ and $\epsilon : \tilde{S}L_2(Z) \rightarrow Z/24$ is the multiplier for the Dedekind eta function.

The third chapter is an exposition of joint work with S. Kudla. In this chapter, the cohomology classes associated by Poincaré duality to the special cycles of the first chapter are related to harmonic forms constructed from the theory of the oscillator representation developed in the second chapter.

In the first section of Chapter III, I construct a theta correspondence or lifting from Siegel modular forms to harmonic forms on locally symmetric spaces of orthogonal groups. This corrrespondence is realized by an integral transform. The kernel of this transform is constructed from a system of Schwartz functions obtained by applying an operator, generalizing the exterior derivative, to the Gaussian. This operator, called the Howe operator in [9] (its formula was suggested by Roger Howe), has many remarkable properties and merits further study.

In the second section of Chapter III, I prove the equality of the two spaces of forms referred to in the paragraph above. The key ingredient in the proof is a formula relating periods of forms on locally symmetric spaces of orthogonal groups to Fourier coefficients of the Siegel modular forms obtained by applying to them the adjoint of the lifting constructed above. This formula, denoted (S) in this paper, is the analogue of the main formulas of Hirzebruch-Zagier [4] and Shintani [18]. It is proved in the last section of this paper.

My results with Kudla together with the recent injectivity results of Rallis [17] hold out the possibility of computing the dimensions of many cohomology groups of orthogonal, unitary and quaternion unitary locally symmetric spaces in terms of spaces of "classical" modular forms in certain tube domains. What remains to be done is to prove surjectivity of our lifting in some cases. This seems to be a very difficult problem but it is perhaps within reach of the powerful new techniques of representation theory.

To conclude, it would appear that the results and methods of this paper generalize to *finite volume* quotients of orthogonal symmetric spaces. Since this is rather surprising I will now sketch how this goes.

The lifting kernel θ_φ is a closed nq form in the orthogonal variable so the lift maps into *closed* harmonic nq forms and accordingly into the absolute cohomology of degree nq. The statement and proof of the Main Theorem must now be modified to use Poincare duality on a non-compact manifold. We now use the pairing, again denoted $[\ ,\ ]$, between nq forms with arbitrary support and compactly supported $(p-n)q$ forms given by (M denotes the locally symmetric space):

$$[\eta, \omega] = \int_M \eta \wedge \omega$$

Now define the Siegel modular form $\theta_\varphi(\eta)$ for η compactly-supported of degree

$(p-n)q$ as $[\eta, \theta_\varphi]$. One finds easily that:

$$(f, \theta_\varphi(\eta)) = [\theta_\varphi(f), \eta]$$

where the inner product on the left is the Petersson inner product on $S_{m/2}(\Gamma')$. Now consider the following two subspaces of the cohomology of degree $(p-n)q$ with compact supports. The first, $H_\theta^\perp$, is the space of all classes of closed compactly supported $(p-n)q$ forms which are orthogonal under $[\ ,\ ]$ to the image of $S_{m/2}(\Gamma')$. The second, $H_{\text{cycle}}^\perp$, is the space of all classes of closed compactly supported $(p-n)q$ forms with period zero over all the special cycles C_β of positive type (see the discussion preceding the Main Theorem). Then it follows immediately from a suitable generalization of the formula (S) that:

$$H_\theta^\perp = H_{\text{cycle}}^\perp$$

Then, by Poincaré duality, the subspace of the absolute cohomology of degree nq spanned by the Poincaré duals of special cycles of positive type coincides with the space of lifts.

I would like to dedicate this paper, a summary of a number of years work, to my parents. I would like to express my gratitude to my collaborator, Steve Kudla, for a considerable amount of help with Chapter III. I would like to thank Dennis Johnson and Bill Dwyer for helpful conversations concerning Chapter II. They made me aware of the significance of quadratic forms over $Z/2$ and the groups Γ_Q. Finally, I have profited greatly from the ideas of Roger Howe.

I. Special cycles in locally symmetric spaces

1. **Locally symmetric spaces.** A Riemannian manifold D is a *symmetric space* if for each x in D there is an isometry θ of D having x as an isolated fixed-point. From this definition it follows that D is homogeneous, so $D = G/K$ with G the group of isometries of D. We assume that G is a non-compact semi-simple Lie group (and K a maximal compact subgroup). Then D is contractible and has infinite volume.

A Riemannian manifold M is a *locally symmetric space* if its universal cover is a symmetric space. From this definition, it follows that M is the quotient of D by a discrete subgroup $\Gamma \subset G$ acting properly discontinuously on D. Thus we may represent M as a double coset space $M = \Gamma\backslash G/K$. The manifold M is a space of type $K(\Gamma, 1)$ and consequently one has:

$$H^*(M;R) = H^*(\Gamma;\mathbb{R}) \tag{I.1.1}$$

$$H^*(M;\tilde{V}) = H^*(\Gamma;V) \tag{I.1.2}$$

where V is an $\mathbb{R}(\Gamma)$-module, $\tilde{V}$ is the associated local coefficient system and the right-hand side of (I.1.1) and (I.1.2) are abstract group cohomology groups.

To give some examples of the type of symmetric and locally symmetric spaces we are considering, we note that they include the hyperbolic plane and its quotients, the Riemann surfaces of genus greater than 1 equipped with the Poincaré metric. Hyperbolic space and its quotients give one generalization of these fundamental examples. Another generalization is the Siegel space and its quotient by $Sp_n(\mathbb{Z})$ which we now describe because it will play an important role in what follows.

We define the symplectic group $Sp_n(\mathbb{R})$ to be the isometry group of the standard skew-symmetric form on $\mathbb{R}^{2n}$; that is, the form $\langle\ ,\ \rangle$ satisfying $\langle e_i, f_j\rangle = \delta_{ij}$ where $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is the standard basis for $\mathbb{R}^{2n}$. We define an almost complex structure to be positive definite if the bilinear form B on $\mathbb{R}^{2n}$ defined by $B(x,y) = \langle x, Jy\rangle$ is symmetric and positive definite. Then the space D of positive definite complex structures on $\mathbb{R}^{2n}$ is the symmetric space for $Sp_n(\mathbb{R})$. Note that B is symmetric if and only if $J \in Sp_n(\mathbb{R})$. From this it is easy to prove that $Sp_n(\mathbb{R})$ acts transitively on D. Moreover the isotropy group of the complex structure J_0 defined by $J_0(e_j) = f_j$, $J_0(f_j) = -e_j$ is $Sp_n(\mathbb{R}) \cap 0(2n)$, a maximal compact subgroup of $Sp_n(\mathbb{R})$ isomorphic to $U(n)$. The quotient of D by $Sp_n(\mathbb{Z})$, the group of integral matrices in $Sp_n(\mathbb{R})$, is a very important space — the moduli space for principally polarized abelian varieties. Another model for the previous symmetric space is the Siegel space $\mathfrak{H}_n$ of n by n complex symmetric matrices $\tau = u + iv$ with imaginary part v positive definite. The complex structure J_0 corresponds to the matrix iI_n where I_n is the n by n identity matrix.

Other generalizations of the hyperbolic plane come from the symmetric spaces of the orthogonal groups. Let V be an n dimensional real vector space with a quadratic form $(\ ,\ )$ of signature (p,q). Let D be the open submanifold of the Grassmannian $G_q(\mathbb{R})$ consisting of the negative q-planes; that is, q-planes Z such that the restriction of $(\ ,\ )$ to Z is negative definite. Then D is a symmetric space and $0(p,q)$, the orthogonal group of $(\ ,\ )$ acts transitively on D. To each negative q-plane Z we may associate the involution r_Z which is -1 on Z and $+1$ on $Z^\perp$ the positive definite quadratic form $(\ ,\ )_Z$ on V defined by:

$$(u,v)_Z = (r_Z u, v).$$

The form $(\ ,\ )_Z$ is said to be a Hermite majorant for $(\ ,\ )$. The subspace m of all positive definite forms on V defined by:

$$m = \{(\ ,\ )_Z : Z \in D\}$$

is called the space of Hermite majorants of $(\ ,\ )$. A positive definite form R is in m if and only if it majorizes $(\ ,\ )$; that is it satisfies $R(x,x) \geq |(x,x)|$ for all x and it is minimal among all majorants. The identification of D and m is at the heart of Siegel's work on theta functions of indefinite quadratic forms.

We now describe in some detail the most general examples considered in [9]. Our later construction of harmonic forms via the Weil representation will apply to these examples. Let $\mathbb{F}$ represent either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the Hamilton quaternions $\mathbb{H}$. Let ι be the involution of $\mathbb{F}$ which is respectively the identity, complex conjugation or quaternionic conjugation. Let $(\ ,\ )$ be a hermitian (relative to ι) form on $\mathbb{F}^m$ of signature (p,q); hence $(\ ,\ )$ maps $\mathbb{F}^m \times \mathbb{F}^m$ to $\mathbb{F}$ and satisfies for $x, y, z \in \mathbb{F}^m$ and $c \in \mathbb{F}$:

(*i*) $$(x + y, z) = (x, z) + (y, z)$$

(*ii*) $$(x, yc) = (x, y)c$$

(*iii*) $$(xc, y) = \iota(c)(x, y)$$

(*iv*) $$(y, x) = \iota((x, y)).$$

As in the previous example, let D be the open submanifold of the Grassmannian $G_q(\mathbb{F})$ consisting of all negative q-planes; that is, q-planes Z such that the restriction of (,) to Z is negative definite. Then D is a symmetric space and the group G of isometries of (,) acts transitively on D. To obtain compact quotients of D one must repeat the above construction taking F to be respectively a real quadratic field $\varkappa = \mathbb{Q}(\sqrt{n})$, a totally imaginary extension of $\varkappa$. We require (,) to be $\varkappa$-valued and to be of signature (p, q) at one completion of $\varkappa$ and positive definite at the other. For example one might take the following forms in the three cases:

(*i*) $$x_1^2 + \dots + x_p^2 - \sqrt{n}(x_{p+1}^2 + \dots + x_m^2)$$

(*ii*) $$|z_1|^2 + \dots + |z_p|^2 + \sqrt{n}(|z_{p+1}|^2 + \dots + |z_m|^2)$$

(*iii*) $$|q_1|^2 + \dots + |q_p|^2 - \sqrt{n}(|q_{p+1}|^2 + \dots + |q_m|^2)$$

Then we choose for Γ a subgroup of finite index in the (algebraic) integral isometries of (,). Then $\Gamma \subset G$, $M = \Gamma \backslash D$ will always be compact and if Γ contains no elements of finite order M will be a manifold. For more details see Borel [1]. We will call these examples, the basic examples.

2. **Differential forms on a symmetric space.** In this section we give convenient representations of differential forms, the exterior derivative and the Hodge Laplacian on a locally symmetric space. Choose a maximal compact subgroup of G (or a point on D). Let g be the Lie algebra of G, l the Lie algebra of K and p the orthogonal complement of l for (,) the Killing form of g. Then the left invariant distribution on G defined by p is the horizontal distribution for the Riemannian connection on M. To be quite precise we should say that the frame bundle of M reduces as a bundle with connection to the bundle $p : \Gamma \backslash G \to \Gamma \backslash G / K$ with the connection described above. Let $\{\omega_i\}$ be a basis for the left-invariant horizontal forms (elements of p^*, the dual of p). If η is a k-form on $\Gamma \backslash G / K$ then $p^*\eta$ may be written in terms of $\omega_{i_1} \wedge \omega_{i_2} \wedge \dots \wedge \omega_{i_k}$ for $i_1 < i_2 < \dots < i_k$ according to $p^*\eta = \sum_I f_I \omega_I$ (here I is a multi-index). Then the system of functions $\{f_I\}$ gives an element $\tilde{\eta} \in (C^\infty(\Gamma \backslash G) \oplus \Lambda^p p^*)^K$. That is, $\tilde{\eta}$ satisfies:

(I.2.1) $$\tilde{\eta}(\gamma g) = \tilde{\eta}(g)$$

(I.2.2) $$\tilde{\eta}(gk) = \sigma(k^{-1})\tilde{\eta}(g)$$

where σ is the natural action of K on $\Lambda^p p^*$. Conversely, every element $\{f_I\} \in (C^\infty(\Gamma \backslash G) \oplus \Lambda^p p^*)^K$ gives rise to a p-form η on M satisfying $p^*\eta = \Sigma f_I \omega_I$. An identical situation holds for sections of any homogeneous bundle E_σ defined by a representation σ of K.

We now record how the exterior derivative d and the Hodge Laplacian Δ carry over to this representation of a differential form. Let $\{E_i\}$ be a basis for the left invariant vector fields which is dual to $\{\omega_i\}$. Then:

$$\tilde{d}\eta = \Sigma E_i \otimes A(\omega_i)\cdot\tilde{\eta}$$

where $A(\omega_i):\Lambda^p p^* \to \Lambda^{p+1} p^*$ is the operator of exterior multiplication by ω_i.

Let $\{X_i : i = 1, \dots, n\}$ be a basis for l satisfying $(x_i, x_j) = -\delta_{ij}$ and $\{Y_j : j = 1, \dots, m\}$ an orthogonal basis for p. Then $C = -\sum_{i=1}^{m} Y_j^2 + \sum_{i=1}^{n} X_i^2$ is a second order differential operator on G which commutes with the left and right actions of G. Hence C induces an operator on $(C^\infty(\Gamma\backslash G) \otimes \Lambda^p p^*)^K$ and we have:

$$\Delta\tilde{\eta} = (C \otimes I)\tilde{\eta} \quad \text{(Kuga's Lemma).} \tag{1.2.3}$$

Finally we will need a formula in terms of $\tilde{\eta}$ for the period of a differential p-form η over a p-dimensional sub-symmetric space $\Gamma_1\backslash G_1/K_1 \subset \Gamma\backslash G/K$. Let $p_1 \subset p$ be the complement to l_1 in g_1. Choose an orthonormal basis for p^* so that the first p basis vectors $\omega_1, \omega_2, \dots, \omega_p$ form an oriented orthonormal basis for p_1^*. Then $p^*\eta = \Sigma\, \eta_I \omega_I$. Let I_0 be the multi-index $\{1, 2, \dots, p\}$. Then we have:

$$\int_{\Gamma_1\backslash G_1/K_1} \eta = \frac{1}{\operatorname{vol} K_1} \int_{\Gamma_1\backslash G_1} \eta_{I_0}. \tag{1.2.4}$$

3. **The cohomology of locally symmetric spaces.** We begin our study of cohomology by noting that every form on D which is invariant under G is harmonic by I.2.3. Thus any non-zero invariant form represents a non-zero cohomology class in $M = \Gamma\backslash D$. Matsushima, Osaka Journal, 1962, proved that the cohomology of "low degree" relative the dimension of M consists entirely of such classes. Thanks to recent work of Borel-Wallach [2] and Zuckerman [21] we now know that $H^i(M, \mathbb{R})$ consists entirely of such classes provided i is less than an important geometric invariant r of D known as the rank — the dimension of a maximal isometrically embedded totally geodesic flat torus. For many (but not all) D there are compact quotients $M = \Gamma\backslash D$ and classes in $H^r(\Gamma\backslash D; \mathbb{R})$ which are not represented by invariant forms. The construction of some such classes is the subject of the next section.

4. **Special cycles in locally symmetric spaces.** Let $\pi : D \to \Gamma\backslash D$ be the projection map in what follows. Let σ_1 be an isometry of order 2 so that $\sigma_1\Gamma\sigma_1 = \Gamma$ and let D_1 be the fixed-point set of σ_1 in D. Let $M_1 = \pi(D_1)$. Then we say M_1 is a *special cycle* in $\Gamma\backslash D$. We first note a very useful lemma which makes special cycles pleasant to deal with. Let $\Gamma_1 = \{\gamma \in \Gamma : \sigma_1\gamma\sigma_1 = \gamma\}$.

LEMMA I.4.1. (Harris Jaffee). *If Γ is torsion free than $\pi(D_1) = \Gamma_1\backslash D_1$.*

PROOF. Suppose x and y in D_1 have the same image under π. Then there exists $\gamma \in$

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Γ so that $\gamma x = y$. Then $\tau = \gamma^{-1}\sigma_1\gamma\sigma_1$ is an element of Γ fixing x. Consequently τ is the identity and $\gamma \in \Gamma_1$. With this the Jaffee lemma is proved.

COROLLARY. *M_1 is itself a locally symmetric space $M_1 = \Gamma_1 \backslash G_1 / K_1$ where G_1 (resp. K_1) is the centralizer of σ_1 in G (resp. K).*

We now make a conjecture which Raghunathan and I proved in many cases.

CONJECTURE. For σ_1, Γ as above there exists a subgroup Γ' of finite index in Γ so that $\pi'(D_1)$ is not a boundary (here $\pi' = D \to \Gamma' \backslash D$ is the projection).

The conjecture seems a reasonable one for several reasons. First, it is consistent with all known vanishing theorems. Second, it has been proved in a large number of cases. Finally every special cycle comes with a complementary special cycle as we now explain. We assume there exist points on D_1, to be called rational points, so that the associated Cartan involutions θ have the property that $\theta\Gamma\theta \cap \Gamma$ has finite index in Γ. This is always the case for the basic examples.

LEMMA 1.4.2. (Millson-Raghunathan [13]). *Every special cycle has a complementary special cycle.*

PROOF. Choose a rational point $x \epsilon D_1$. Then the reflection θ at x has the property that $\theta\Gamma\theta \cap \Gamma$ has finite index in Γ. We consider the isometry $\sigma_2 = \theta\sigma_1$. We note $\sigma_2^2 = \theta\sigma_1\theta\sigma_1$ fixes x and the tangent space to D at x. Hence σ_2 is an involution and $\sigma_2\Gamma\sigma_2 \cap \Gamma$ has finite index in Γ. By replacing Γ by a subgroup of finite index in Γ we may assume θ and σ_2 normalize Γ. Consequently, if D_2 denotes the fixed-point set of σ_2 in D then $M_2 = \pi(D_2)$ is compact. Clearly it has dimension complementary to that of M_1 and the lemma is proved.

We remark that D_1 and D_2 intersect only at z. However, M_1 and M_2 can have many intersections corresponding to $\gamma \epsilon \Gamma$ so that $\gamma D_1 \cap D_2 \neq \emptyset$.

In [13], Raghunathan and I studied the intersection number of M_1 and M_2. We were able to prove in many cases that it was non-zero. We also proved that if the intersection number was non-zero then there was a finite cover of M so that the Poincaré dual of any component of the inverse image of M_1 could not be represented by an invariant form.

5. **Linear cycles in the basic examples**. For the basic examples, the special cycles have a geometric interpretation in terms of the linear structure of the Grassmannian which makes them appear to be the analogue of certain Schubert cycles.

We choose a $\varkappa$-rational orthogonal splitting $\mathbb{F}^n = X \oplus X^\perp$ where X has $\mathbb{F}$ dimension k and signature (r, s) at the indefinite completion of (,). Thus $X^\perp$ has $\mathbb{F}$ dimension $n - k$ and signature $(p - r, q - s)$. Let σ_1 be the involution which has $+1$ eigenspace equal to X and -1 eigenspace equal to $X^\perp$. Then D_1 is the set of negative q-planes Z which are compatible with the above splitting in the sense that

$$Z = Z \cap X + Z \cap X^\perp.$$

We will denote this subspace D_X. To get the complementary cycle choose a rational q-plane Z_0 in D. Then we have a splitting $\mathbb{F}^n = Z_0 \oplus Z_0^\perp$ and θ, the reflection at Z_0 is represented by the linear transformation which is $+1$ on Z_0 and -1 on $Z_0^\perp$. We find that D_2 is the set of negative q-planes compatible with the splitting $\mathbb{F}^n = U \oplus U^\perp$ where $U = X \cap Z_0 + X^\perp \cap Z_0^\perp$ and $U^\perp = X \cap Z_0^\perp + X^\perp \cap Z_0$. Note that the above decomposition of $\mathbb{F}^n$ is just the eigenspace decomposition of $\mathbb{F}^n$ relative to $\sigma_2 = \theta\sigma_1$. The real dimension of D is pq, $2pq$ or $4pq$ in three cases and the dimension of D_1 is $rs + (p - r)(q - s)$, $2[rs + (p - r)(q - s)]$ or $4[rs + (p - r)(q - s)]$ in the three cases.

The most important special case of the above construction is the case in which X is positive definite. In this case $Z \cap X = \{0\}$ and consequently $D_1 = \{Z \in D : Z \subset X^\perp\}$. We will say cycles obtained from such X have positive type. If X is negative definite we say D_1 has negative type. Finally we will say other cycles (where X is indefinite) have mixed type. We will often write D_X for D_1 and C_X for $\pi(D_X)$. In case X is positive and x is a spanning set for X, we will write D_x and C_x in place of D_X and C_X.

The considerable difference between the cycles of pure type and of mixed type is illustrated in the case $G = SO(2, 2)$. Then D is a product of two hyperbolic planes. The cycles of pure type give rise to algebraic curves – Hirzebruch-Zagier cycles [4], whereas those of mixed type give rise to totally real tori. They are transcendental cycles and have been used by Oda [15].

In case $\mathbb{F} = \mathbb{R}$ and X is a positive 1 dimensional space then D_1 is the set of all negative q-planes perpendicular to a fixed line. If q is even, then Millson-Raghunathan [13] proved that there are uniform $\Gamma \subset SO(p, q)$ so that the cycles M_1 and M_2 had non-zero intersection number. Consequently there is a class in $H^q(\Gamma \backslash D; \mathbb{R})$ which is not represented by an invariant form. Since the rank of D is q, this implies that the Borel-Wallach, Zuckerman theorem is best possible for orthogonal groups of even rank. For subgroups of $O(n, 1)$ there is a very simple geometric argument proving that $H^1(\Gamma \backslash D; \mathbb{R}) \neq \{0\}$ – see Millson [12]. By Rallis [17], we now know that the Borel-Wallach, Zuckerman theorem is best possible for all orthogonal groups.

II. The oscillator representation and theta functions

1. **The existence of the oscillator representation.** Let W be a vector space over $\mathbb{R}$ of dimension $2n$ and let $\langle\ ,\ \rangle$ by a symplectic form on W. Define the Heisenberg group $H(W)$ by:

$$H(W) = W \times S^1 \text{ as a set}$$

and has group law

$$(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, e^{i\pi\langle w_1, w_2\rangle} t_1 t_2)$$

Thus $H(W)$ is a group extension $S^1 \to H(W) \to W$ whose characteristic class is represented relative the above product decomposition by the cocyle:

$$\alpha(w_1, w_2) = e^{i\pi\langle w_1, w_2\rangle}$$

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Let χ be the identity character of S^1 and $\lambda = 2\pi i$ (it is convenient to retain λ as a parameter). Then we have the famous theorem:

Stone-von Neumann theorem. *There is one and only one equivalence class of irreducible unitary representations of $H(W)$ with central character χ.*

We will denote this class by ρ_χ. We now give a realization of ρ_χ. Let F be a Lagrangian, i.e. maximal isotropic, subspace of W. We consider the abelian subgroup $N = F \times S^1$ of $H(W)$. We extend χ to a character of N by making it trivial on F. Then $\operatorname{ind}_N^{H(W)}\chi$ is a unitary representation of $H(W)$. By definition the space of this representation is the space of functions Φ on $H(W)$ satisfying:

(*i*) $$\Phi(xn) = \chi(n)^{-1}\Phi(x)$$

(*ii*) $$|\Phi|^2 \text{ is square integrable on } H(W)/N$$

We denote the space of such functions Φ by $W(F)$. Then $H(W)$ acts on $W(F)$ by left translation. The following theorem is proved in Lion-Vergne [11], page 19.

THEOREM II.1.1 *$W(F)$ is irreducible.*

REMARK. $W(F)$ is called the Schrödinger model of ρ_χ. Let E be a Lagrangian complement to F.

Clearly any function in $W(F)$ is determined by its restriction to E and any square integrable function on E may be extended in exactly one way to an element of $W(F)$. We may accordingly realize the Schrödinger model on $L^2(E)$ with $H(W)$ acting by twisted translations; that is, if $\varphi \in L^2(E)$ then:

$$\rho_\chi(y)\varphi(x) = e^{\lambda(y,x)}\varphi(x) \text{ for } y \in F$$

$$\rho_\chi(x_0)\varphi(x) = \varphi(x - x_0) \text{ for } x_0 \in E$$

$$\rho_\chi(t)\varphi(x) = t\varphi(x)$$

We choose a symplectic basis $\{e_1, e_2, \ldots, e_n; f_1, f_n\}$ for W compatible with the splitting $W = E \oplus F$. We define coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ on W dual to the previous basis. The infinitesmal action of the Lie algebra of $H(W)$ is then given by:

$$d\rho_\chi(e_j) = -\frac{\partial}{\partial q_1}$$

$$d\rho_\chi(f_j) = -\lambda q_j$$

$$d\rho_\chi(\eta) = \lambda I$$

where we have written the Lie algebra n of the Heisenberg group as $W \oplus \mathbb{R}\eta$.

Let $Sp(W)$ be the isometry group of $\langle\ ,\ \rangle$. The action $g\cdot(w,t) = (g(w),t)$ for $g \in Sp(W)$ and $(w,t) \in H(W)$ embeds $Sp(W)$ into the automorphism group of $H(W)$. We will identify $Sp(W)$ with $Sp_n(\mathbb{R})$ using the above basis.

We choose a realization H_χ for the space of ρ_χ. Since $Sp(W)$ acts on $H(W)$ we obtain a new representation ρ_χ^g on H_χ for each $g \in Sp(W)$ defined by $\rho_\chi^g(h) = \rho_\chi(g(h))$. But $Sp(W)$ leaves the center of $H(W)$ fixed, consequently, the central characters of ρ_χ and ρ_χ^g are the same. Then by the Stone-von Neumann theorem there is a unitary operator $\omega_\chi(g)$ satisfying:

$$\omega_\chi(g)\rho_\chi(h)\omega_\chi(g)^{-1} = \rho_\chi(g(h)).$$

The mapping $g \to \omega_\chi(g)$ is easily seen to be a projective representation of $Sp(W)$: that is, it is a homomorphism into the unitary group modulo scalars which we denote $PU(H_\chi)$. Then we have the following diagram:

$$\begin{tikzcd}
S^1 \arrow[r, "id"] \arrow[d] & S^1 \arrow[d] \\
\overline{Sp}(W) \arrow[r, "\omega_\chi"] \arrow[d] & U(H_\chi) \arrow[d] \\
Sp(W) \arrow[r, "\omega_\chi"] & PU(H_\chi)
\end{tikzcd}$$

where $\overline{Sp}(W)$ is the "pull-back" extension under ω_χ of the natural extension of $PU(H_\chi)$.

Let $\tilde{Sp}(W)$ denote the commutator subgroup of $\overline{Sp}(W)$. Since $Sp(W)$ is perfect we have a surjective mapping π: $\tilde{Sp}(W) \to Sp(W)$.

LEMMA II.1.1. *The kernel of* π *is finite.*

PROOF: Let l be the Lie algebra of the circle, sp (resp. $\overline{sp}$) the Lie algebra of $Sp(W)$ (resp. $\overline{Sp}(W)$). Then we have $l \to \tilde{sp} \to sp$. Since sp is simple, this extension splits and since l is one dimensional and sp has no characters the action of sp on l is trivial. Thus $\overline{sp} = sp \oplus l$ and π is an isomorphism of Lie algebras. Thus the kernel of π is a discrete subgroup of S^1 and consequently is finite.

We now examine more closely the infinitesimal oscillator representation $d\omega_\chi$: $sp \to \operatorname{End} H_\chi$. The algebra sp acts on n, the Lie algebra of $H(W)$, by differentiating the action of $Sp(W)$ on HW). We write this action as $[x, y]$ for $x \in sp$, $y \in n$.

LEMMA II.1.2. $d\omega_\chi$ *is the unique Lie algebra homomorphism taking* $sp \to \operatorname{End} H_\chi$ *and satisfying:*

$$d\rho_\chi([x, y]) = [d\omega_\chi(x), d\rho_\chi(y)] \text{ for all } x \in sp,\ y \in n.$$

PROOF: By the previous analysis $d\omega_\chi$ satisfies the above identity. If T is any other homomorphism then $d\omega_\chi - T$ must be a scalar homorphism as $d\rho_\chi$ is irreducible. But sp is equal to its commutator sublagebra and the lemma is proved.

Now let Δ be the kernel of π. Then we have:

$$\begin{tikzcd}
\Delta \arrow[r] \arrow[d] & S^1 \arrow[r] \arrow[d] & S^1 \arrow[d] \\
\tilde{Sp}(W) \arrow[r] \arrow[d] & \overline{Sp}(W) \arrow[r] \arrow[d] & U \arrow[d] & * \\
Sp(W) \arrow[r] & Sp(W) \arrow[r] & PU
\end{tikzcd}$$

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Thus, ω_χ lifts to a linear representation of $\tilde{Sp}(W)$.

We now wish to determine Δ. For any finite subgroup $\Delta \subset S^1$ we will say the extension $S^1 \to \overline{Sp}(W) \to Sp(W)$ is reducible to Δ if there is a group $\tilde{Sp}(W)$ so that the diagram * holds. The following two lemmas allow us to study the same problem for the extension induced over any subgroup $H \subset Sp(W)$ provided $\pi_1(H)$ maps onto $\pi_1(Sp(W))$.

Lemma II.1.3. *The extension* $S^1 \to \overline{Sp}(W)$ *is reducible to* Δ *if and only if the quotient extension* $S^1/\Delta \to Sp(W)/\Delta \to Sp(W)$ *is trivial.*

Proof: We denote the two maps in the diagram * by $i: \tilde{Sp}(W) \to \overline{Sp}(W)$ and $p: \overline{Sp}(W) \to \overline{Sp}(W)/\Delta$. If $\tilde{Sp}(W)$ exists then $p \circ i(\tilde{Sp}(W))$ splits the above extension and if σ is a section then defining $\tilde{Sp}(W) = p^{-1}(\sigma(Sp(W)))$ gives the diagram *.

Lemma II.1.4. *Let* G *be a subgroup satisfying* $\pi_1(G) \to \pi_1(Sp(W))$ *is onto. Then the restriction of* $S^1 \to \overline{Sp}(W) \to Sp(W)$ *to* G *reduces to* Δ *if and only if the original extension* $S^1 \to \overline{Sp}(W) \to Sp(W)$ *reduces to* Δ.

Proof: We have a diagram:

$$\begin{tikzcd}
S^1/\Delta \arrow[r] \arrow[d] & S^1/\Delta \arrow[d] \\
\overline{G}/\Delta \arrow[r] \arrow[d] & \overline{Sp}(W)/\Delta \arrow[d] \\
G \arrow[r] & Sp(W)
\end{tikzcd}$$

Since $\pi_1(G)$ maps onto $\pi_1(Sp(W))$, the triviality of the left-hand extension implies that of the right by Shapiro [18]. This proves the "only if" part of the lemma. The converse is obvious.

We now study $d\omega_\chi$. It will be useful to identify the space of symmetric 2-tensors on W, to be denoted S^2W with sp. This is done by the mapping $S^2W \to \text{End } W$ which associates to the symmetric product $w_1 \circ w_2 = w_1 \otimes w_2 + w_2 \otimes w_1$ in S^2W the endomorphism sending $v \in W$ to $\langle w_1, v\rangle w_2 + \langle w_2, v\rangle w_1$.

There is a remarkably simple way to obtain the infinitesimal oscillator representation. We define the Weyl algebra $\mathcal{W}$ to be the quotient of the tensor algebra $T(W)$ by the ideal generated by $\{w_1 \otimes w_2 - w_2 \otimes w_1 - \lambda\langle w_1, w_2\rangle 1\}$. Then $\mathcal{W}$ is isomorphic to the quotient of $\mathcal{U}(n)$, the universal enveloping algebra of n by the ideal generated by $\{n - \lambda 1\}$. It is also isomorphic under $d\rho_\chi$ to the algebra of polynomial coefficient differential operators on F. $\mathcal{W}$ is an associative algebra and hence has a Lie algebra structure with $[x, y] = xy - yx$. The algebra $\mathcal{W}$ inherits a filtration $\{\mathcal{W}^i\}$ from the filtration on $\mathcal{U}(n)$. Then $\mathcal{W}^2$ is a sub-algebra and $\mathcal{W}^1 = n = W \oplus \mathbb{R}$ is an ideal. It is clear that $\mathcal{W}^2/\mathcal{W}^1$ is isomorphic to S^2W. Moreover we have a canonical linear section $j: S^2W \to \mathcal{W}^2$ given by $j(x \circ y) = \frac{1}{2\lambda}(x \otimes y + y \otimes x)$. The image of j is stable for the bracket operation on $\mathcal{W}$ and j becomes an algebra isomorphism between $sp(W)$ and $j(S^2W)$. Also if $s \in S^2W$ the action of $j(s)$ on $\mathcal{W}^1$ is the same as the usual action of

$sp(W)$ on $n = W \oplus \mathbb{R}$.

Now $d\rho_\chi$ extends to $\mathcal{W}$ and of course satisfies for $s \in S^2W$, $x \in n$:

$$d\rho_\chi([j(s), x]) = [d\rho_\chi(j(s)), d\rho_\chi(x)]$$

By Lemma II.1.5 we have $d\rho_\chi(j(s)) = d\omega_\chi(s)$. Noting that if $s = x \circ y$ we have $j(s) = \frac{1}{2\lambda}(x \otimes y + y \otimes x)$ and hence

$$d\rho_\chi(j(s)) = \frac{1}{2\lambda}(d\rho_\chi(x)d\rho_\chi(y) + d\rho_\chi(y)d\rho_\chi(x))$$

we obtain a useful formula.

LEMMA II.1.5.

$$d\omega_\chi(x \circ y) = \frac{1}{2\lambda}(d\rho_\chi(x)d\rho_\chi(y) + d\rho_\chi(y)d\rho_\chi(x))$$

This formula may be loosely stated as follows: we obtain the infinitesmal oscillator representation by extending the infinitesmal Stone-von Neumann representation to the quadratic elements in the Weyl algebra so that anti-commutators are preserved.

We can now compute the finite group Δ recalling now that $\lambda = 2\pi i$. Indeed by Lemma II.1.4 it is enough to consider the metaplectic extension pulled all the way back to the circle $SO(2)$ which is the maximal compact subgroup of $Sp(P)$ where $P \subset W$ is the plane spanned by $\{e_1, f_1\}$. We identify $\mathbb{R}$ with the Lie algebra of $SO(2)$ by mapping t to $(-t/2)(e_1^2 + f_1^2)$ (the element $(-1/2)(e_1^2 + f_1^2)$ in $S^2(P)$ corresponds to the element J of $Sp(P)$ satisfying $J(e_1 = f_1, J(f_1) = -e_1)$. Then the exponential map from $\mathbb{R}$ to $SO(2)$ is given by $\exp(t) = e^{2\pi t J}$. Thus, the kernel of the exponential map is generated by $(-1/2)(e_1^2 + f_1^2)$. By Lemma II.1.5, this element acts by $(-1/4\pi i)((\partial^2/\partial q_1^2) - 4\pi^2 q_1^2)$. The eigenfunctions of this operator on the Schwartz space are the Hermite functions and the eigenvalues are half-integer multiples of i. Thus, $e^{4\pi J}$ acts trivially and $d\omega_\chi|\mathbb{R}$ descends to a representation of the 2-fold cover of $SO(2)$. We obtain the famous theorem of Segal and Shale.

THEOREM. *The projective representation ω_χ lifts to a linear representation of the 2-fold cover of $Sp(W)$.*

REMARK. We will use the symbol ω_χ for both the projective representation of $Sp(W)$ and the linear representation of the 2-fold cover $\widetilde{Sp}(W)$. The representation ω_χ is che oscillator (or Weil) representation. $\widetilde{Sp}(W)$ is called the metaplectic group and will usually be denoted $Mp(W)$. The extension $Mp(W) \to Sp(W)$ will be called the metaplectic extension.

Let P be a symplectic plane and $G(P)$ the two element subgroup of $Sp(W)$ consisting of the identity and the element ι which is -1 on P and $+1$ on the orthogonal complement of P in W. Then we have the following lemma.

LEMMA II.1.6. *The metaplectic extension restricted to $G(P)$ is not trivial.*

PROOF: We have just seen that the two-fold cover of the circle $SO(2) \subset Sp(P)$ induced by the metaplectic extension is the non-trivial 2-fold cover of the circle by itself. But the inclusion $G(P) \subset SO(2)$ is just the usual inclusion $\{\pm 1\} \subset S^1$. The lemma follows.

COROLLARY. *The metaplectic extension remains non-trivial upon restriction to any subgroup which contains any $G(P)$ (for example the integral symplectic group or its congruence subgroup of level 2).*

2. **Some elements in the metaplectic group**. In the next section we will need to make computations with certain elements of $Mp(W)$ which we define in the paragraphs below. We will use the notation that if $\tilde{G}$ is a covering group of a group G and $\tilde{g} \in \tilde{G}$ then the image of $\tilde{g}$ in $\tilde{G}$ will be denoted g. Let P be the subgroup of $Sp(W)$ which is the stabilizer of F. Let N be the subgroup of P consisting of those elements acting trivially on F and M be the stabilizer of E and F. Then $P = M \cdot N$. Let P^0 denote the identity component of P.

LEMMA II.2.1. *The metaplectic extension restricted to P^0 is trivial.*

PROOF. The lemma will follow if we can show that the mapping on fundamental groups $\pi_1(P^0) \to \pi_1(Sp(W))$ is trivial. But P^0 has $SO(n)$ as a deformation retract and $Sp(W)$ has $U(n)$ as a deformation retract. Thus the above inclusion is equivalent to $\pi_1(SO(n)) \to \pi_1(U(n))$. But the inclusion from $SO(n)$ into $U(n)$ factors through $SU(n)$ which is simply-connected.

Let s be a continuous section of the induced extension of P^0. We recall that the metalinear group $ML_n(\mathbb{C})$ is defined to be the fiber product of the double cover of $\mathbb{C}^*$ and the determinant map, $\det: GL_n(\mathbb{C}) \to \mathbb{C}^*$. Hence, we have the fiber square:

$$\begin{tikzcd}
ML_n(\mathbb{C}) \arrow[r] \arrow[d] & \mathbb{C}^* \arrow[d] \\
GL_n(\mathbb{C}) \arrow[r] & \mathbb{C}^*
\end{tikzcd}$$

The upper horizontal arrow we denote $\sqrt{\det}$ for it satisfies $(\sqrt{\det \tilde{g}})^2 = \det g$. We get corresponding extensions of $U(n)$ and $GL_n(\mathbb{R})$ denoted $MU(n)$ and $ML_n(\mathbb{R})$ respectively. We note that if $m \in GL_n^+(\mathbb{R})$, that is $\det m > 0$, then $\sqrt{\det s(m)} > 0$. We denote $s(GL_n^+(\mathbb{R}))$ by $ML_n^+(\mathbb{R})$. We will identify $GL_n^+(\mathbb{R})$ and $ML_n^+(\mathbb{R})$ using s.

We now construct some elements in $s(P^0)$. We let $X_{ij}(\lambda)$ for $\lambda \in \mathbb{R}$ and $1 \leqq i, j \leqq n$ with $i \neq j$ denote the matrix in P^0 with λ in the i,j position, $-\lambda$ in the $n+j$, $n+i$ position, 1's along the diagonals and zeroes elsewhere. Let $Y_{ij}(\lambda)$ for $1 \leqq i < j \leqq n$ denote the matrix in P^0 with λ in the i, $n+j$ position and λ in the j, $n+i$ position, 1's along the diagonal and zeroes elsewhere.

Let $Y_{ii}(\lambda)$ for $1 \leqq i \leqq n$ denote the matrix in P^0 with λ in the i, $n+i$ position, 1's along the diagonal and zeroes elsewhere. Finally let $\tilde{X}_{ij}(\lambda) = s(X_{ij}(\lambda))$, $\tilde{Y}_{ij}(\lambda) = s(Y_{ij}(\lambda))$ and $\tilde{Y}_{ii}(\lambda) = s(Y_{ii}(\lambda))$. We note, since s is a homomorphism, any relation among X_{ij}, Y_{ij} and Y_{ii} gives rise to the corresponding relation among $\tilde{X}_{ij}$, $\tilde{Y}_{ij}$ and $\tilde{Y}_{ii}$. We also need the matrices in $Sp_n(\mathbb{R})$ given by: $n(u) = \begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix}$ where u is a symmetric n by n matrix, $a(v) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$ where v is an invertible n by n matrix, and $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$.

We also let $\tilde{n}(u) = s(n(u))$, $\tilde{a}(v) = s(a(v))$, and $\tilde{J}$ be the matrix in the connected component of the inverse image of $SU(n)$ in $Mp_n(\mathbb{R})$ which covers J. We put ${}^t\tilde{n}(-u) = \tilde{J}n(u)\tilde{J}^{-1}$. Hence ${}^t\tilde{n}(u)$ covers ${}^tn(u)$.

LEMMA II.2.2. *We have the following relations in* $\widetilde{Sp}_n(\mathbb{Z})$:

$$[\tilde{X}_{ij}(1), \tilde{Y}_{ij}(1)] = \tilde{Y}_{ii}(2) \qquad \text{(R1)}$$

$$[\tilde{X}_{ij}(1), \tilde{Y}_{jj}(2)] = \tilde{Y}_{ii}(2)\,\tilde{Y}_{ij}(2) \qquad \text{(R2)}$$

PROOF: The corresponding relations for X_{ij}, Y_{ij}, Y_{ii} may be verified by matrix multiplication or by observing that they are Steinberg relations for $Sp_n(\mathbb{R})$. The above relations follow upon applying the section s.

3. **Splitting the metaplectic extension, the theta distribution and $\boldsymbol{\kappa}^{1/2}$.** We now choose a lattice $L \subset W$ so that $\langle\ ,\ \rangle$ is integral on L and is unimodular. We take up the question of the behaviour of the metaplectic extension upon restriction to discrete subgroups $\Gamma \subset Sp(L)$ where $Sp(L)$ denotes the subgroup of $Sp(W)$ which stabilizes L. We have induced extensions $\mathbb{Z}/2 \to \tilde{\Gamma} \to \Gamma$ and $S^1 \to \bar{\Gamma} \to \Gamma$ which we denote $*$. We recall our observation of the previous section that to split $*$ it is necessary and sufficient to lift $\omega_\chi|\Gamma$ to a linear representation of Γ.

We will construct a concrete model for ρ_χ and ω_χ called the lattice model. The inclusion $L \subset W$ allows us to form an extension $S^1 \to \mathbb{R} \to L$.

LEMMA II.3.1. *R is a maximal abelian subgroup of $H(W)$.*

PROOF: We have the following formula for commutators in $H(W)$:

$$[(w_1, t_1), (w_2, t_2)] = (0, e^{2\pi i\langle w_1, w_2\rangle}).$$

Thus if w_1, $w_2 \in L$ then (w_1, t_1) and (w_2, t_2) commute since $\langle\ ,\ \rangle$ is integral on L. Also if (w, t) commutes with $(\ell, 1)$ for all $\ell \in L$ then $\langle w, \ell\rangle$ is an integer for all $\ell \in L$. Consequently $w \in L$ and the lemma is proved.

COROLLARY: *$S^1 \to R \to L$ is trivial.*

PROOF: L is a free abelian group.

Now let Q be a quadratic form on $L/2L$ with values in $\mathbb{Z}/2$ associated to $\langle\ ,\ \rangle$ (this means Q satisfies $Q(x + y) = Q(y) + Q(y) + \langle x, y\rangle$). We define a function χ_Q from R to S^1 by:

$$\chi_Q(\ell, t) = e^{i\pi Q(\bar{\ell})}t \text{ (here } - \text{ denotes reduction modulo 2)}$$

Lemma II.3.2. *χ_Q is a character.*

Proof: The proof is evident.

We now consider the representations $\mathrm{ind}_R^{H(W)}\chi_Q$. We denote the Hilbert space of this representation by $\mathcal{L}_\chi$ and the representation (temporarily) by ρ. We note that we may take for $\mathcal{L}_\chi$ the space of functions F on $H(W)$ satisfying:
(i) $F(xr) = \chi_Q(r)^{-1}F(x)$
(ii) $|F|^2$ is square integrable on $H(W)/R$. Then $H(W)$ acts by left translations.

Clearly any element of $\mathcal{L}_\chi$ is determined by its restriction to W (identified with $(w, 1) \in H(W)$). We find that the image of $\mathcal{L}_\chi$ under this restriction mapping is the space of functions for W saitsfying for $\ell \in L$ and $w \in W$:
(i) $f(w + \ell) = e^{i\pi\langle \ell, w\rangle}e^{-i\pi Q(\bar{\ell})}f(w)$
(ii) $\int_{W/L}|f(w)|^2dw$ is finite
The action of $W \subset H(W)$ on this space is now by twisted translations:

$$\rho(w)\cdot f(w') = e^{-i\pi\langle w, w'\rangle}f(w' - w)$$

The functions f are (non-holomorphic) analogues of theta functions.

Lemma II.3.3. *ρ is irreducible.*

Proof: We show that the only bounded operators commuting with ρ are the scalar multiples of the identity. We note:

$$\rho(r)F(x) = F(r^{-1}x) = F(xr^{-1}[r^{-1},x]) = \chi_Q(r)\chi_Q([x,r^{-1}])F(x)$$

Thus $r \in R$ acts by the multiplication operators $\chi_Q(r)\chi_Q([x,r^{-1}])$. We claim such functions separate the points of $H(W)/R$. Indeed if $x_1 = (w_1, t_1)$ and $x_2 = (w_2, t_2)$ then $\chi_Q(r)\chi_Q([x_1,r^{-1}]) = \chi_Q(r)\chi_Q([x_2,r^{-1}])$ for all $r \in R$ implies $[x_1,r^{-1}] = [x_2,r^{-1}]$ for all $r \in R$. But by the commutator formula of Lemma II.2.1 this implies $e^{2\pi i\langle w_1, \ell\rangle} = e^{2\pi i\langle w_2, \ell\rangle}$ for all $\ell \in L$ which implies $w_1 - w_2 \in L$ and $x_1R = x_2R$.

We find then that any bounded operator M on $\mathcal{L}_\chi$ commuting with all $\{\rho(r): r \in R\}$ commutes with all multiplication operators by continuous functions on $H(W)/R$ hence is given (in the second model above) by $Mf(x) = m(x)f(x)$ where $m(x)$ is a bounded function on W/L. From the identity $\rho(w) \circ M \circ \rho(w)^{-1} = M$ we find $m(x + w) = m(x)$ for all w. Consequently m is constant and the lemma is proved.

Thus ρ is a realization of the Stone-von Neumann representation of $H(W)$. It is called the lattice model. We should emphasize that the space $\mathcal{L}_\chi$ depends on the choice of quadratic form Q. Of course we have a realization of the oscillator representation ω_χ on $\mathcal{L}_\chi$. Let $O(Q)$ denote the orthogonal group of Q so $O(Q) \subset Sp(L/2L)$. Let Γ_Q denote the inverse image in $Sp(L)$ of $O(Q)$ under the canonical mapping $Sp(L) \rightarrow Sp(L/2L)$. We have the following theorem.

Theorem II.3.1. *The projective representation $\omega_\chi: Sp(W) \rightarrow PU(\mathcal{L}_\chi)$ becomes a linear representation when restricted to Γ_Q for Q any non-singular quadratic form over $\mathbb{Z}/2$ associated to $\langle\ ,\ \rangle$.*

PROOF. We first note that if we form the lattice model ρ_χ, with the character χ corresponding via Lemma II.2.2 to the quadratic form Q, then Γ_Q preserves the inducing data used to define ρ_χ. Thus Γ_Q acts on $\mathscr{L}_\chi$ by $A:\Gamma_Q \to U(\mathscr{L}_\chi)$ given by:

$$A(\gamma)f(x) = f(\gamma^{-1}(x)).$$

It is immediate that:

$$A(\gamma)\rho_\chi(n)A(\gamma^{-1}) = \rho_\chi(\gamma(n)).$$

Hence $A(\gamma)$ represents $\omega_\chi(\gamma)$ and the theorem is proved.

COROLLARY. *The S^1 extension associated to the metaplectic extension of Γ_Q is trivial.*

We now apply the previous theorem to obtain a formula for the characteristic class σ of the extension $\mu_2 \to \tilde{\Gamma}_Q \to \Gamma_Q$. We consider the long exact cohomology sequence attached to the coefficient sequence $\mu_2 \to S^1 \to S^1$, where the second map is squaring. The part we are interested in is:

$$H^1(\Gamma_Q, S^1) \xrightarrow{\delta} H^2(\Gamma_Q, \mu_2) \xrightarrow{j} H^2(\Gamma_Q, S^1)$$

(μ_n will be used to denote the nth roots of unity).

We have seen that $j(\sigma)$ is trivial, consequently, by exactness, there exists a character μ of Γ_Q so that $\delta\mu = \sigma$. Let us examine this more closely in terms of representing chains. Let c by any μ_2 valued cocyle representing σ. Then there exists an S^1 valued 1-cochain α on Γ_Q so that $c(g_1, g_2) = \alpha(g_1 g_2)\alpha(g_2)^{-1}\alpha(g_1)^{-1}$. Since c takes values in μ_2 we see that α^2 is a character of Γ_Q and $\delta\alpha^2 = -c$.

We now give a formula for μ in terms of the oscillator representation. The oscillator representation is a linear representation $\omega_\chi : \tilde{\Gamma}_Q \to U(\mathscr{L}_\chi)$. Thus there exists a character $\epsilon : \tilde{\Gamma}_Q \to S^1$ so that:

$$\omega_\chi(\gamma) = \epsilon(\gamma)A(\gamma)$$

Let Θ denote the linear functional on the smooth functions of $\mathscr{L}_\chi$ which is the Dirac delta at the identity of $H(W)$. Then since $A(\gamma)$ leaves Θ invariant we have:

$$\omega_\chi(\gamma)\Theta = \epsilon(\gamma)^{-1}\Theta$$

We observe that ϵ^2 annihilates the center of $\tilde{\Gamma}_Q$ and consequently induces a character, to be denoted λ, of Γ_Q. Then $\lambda(\gamma) = \epsilon^2(s(\gamma))$ where $s(\gamma)$ is any element of $\tilde{\Gamma}_Q$ lying over γ.

Now let $s:\Gamma_Q \to \tilde{\Gamma}_Q$ be a set-theoretic cross-section chosen so that $c(g_1, g_2) = s(g_1 g_2)s(g_2)^{-1}s(g_1)^{-1}$ is a μ_2 valued cocycle representing the class §. Applying ω_χ and operating on the theta distribution we find $c(g_1, g_2) = \epsilon(s(g_1 g_2))\epsilon(s(g_2))^{-1}\epsilon(s(g_1))^{-1}$. Applying the previous discussion with $\alpha(g) = \epsilon(s(g))$ and noting $\alpha^2(g) = \epsilon^2(s(g)) = \lambda(g)$ we find $\delta\lambda$ represents §.

We obtain the following formula for a representative cocycle c for the metaplectic extension of Γ_Q in terms of the theta multiplier ϵ. For each $\gamma \in \Gamma_Q$, let $\sqrt{\lambda}(\gamma)$ be one of the two square roots of $\lambda(\gamma)$. Then, we have the following lemma.

LEMMA II.3.4. *The metaplectic class restricted to Γ_Q is represented by:*

$$c(\gamma_1, \gamma_2) = \delta\lambda(\gamma_1, \gamma_2) = \frac{\sqrt{\lambda(\gamma_1\gamma_2)}}{\sqrt{\lambda(\gamma_1)}\sqrt{\lambda(\gamma_2)}}$$

COROLLARY: *If Λ' is any subgroup of $\tilde{\Gamma}_Q$ which leaves Θ fixed and Λ is the image of Λ' in Γ_Q then the restriction of the metaplectic extension to Λ is trivial.*

REMARKS: An equivalent statement of the previous lemma is that the extension $\tilde{\Gamma}_Q \to \Gamma_Q$ is the pull-back via λ of the 2-fold extension of the circle. We note that the corollary is obvious directly because Λ' cannot contain the center of $\tilde{\Gamma}_Q$ (which acts by -1 on Θ).

We now construct Λ'. We assume Q is the form given by:

$$Q(x_1, x_2, \ldots, x_{2n}) = \sum_{i=1}^{n} x_i x_{n+i}$$

In this case Γ_Q consists of integral symplectic matrices such that ${}^t ac$ and ${}^t bd$ have even diagonal entries and is known as the *theta group*.

LEMMA II.3.5. *For $n > 1$, the character ϵ is trivial on the subgroup of $\tilde{\Gamma}_Q$ generated by $\tilde{n}(2u)$ and ${}^t\tilde{n}(2u)$ where u is an n by n symmetric integral matrix. (See the end of the previous section for the meaning of the notation in this lemma).*

PROOF. We claim that the matrix $\tilde{n}(2u)$ is a product of commutators in $\tilde{\Gamma}_Q$. Since $\tilde{n}(u_1)\tilde{n}(u_2) = \tilde{n}(u_1 + u_2)$, it is enough to verify the claim for $\tilde{Y}_{ij}(2)$ and $\tilde{Y}_{ii}(2)$. But the relation (R1) of Lemma II.2.2 implies $\tilde{Y}_{ii}(2)$ is a commutator, then by applying the relation (R2) we see that $\tilde{Y}_{ij}(2)$ is a product of two commutators. Since ${}^t\tilde{n}(2u)$ is conjugate to $\tilde{n}(2u)$ (note $J \in \Gamma_Q$) the lemma follows.

It is proved in Mumford [14], Proposition A1, that the elements $Y_{ij}(2)$ and $Y_{ii}(2)$ (for $1 \leqq i, j \leqq n$) generate the subgroup of $Sp_n(Z)$ of matrices satisfying the congruence conditions; $a \equiv d \equiv 1_n \bmod 4$, $b \equiv c \equiv 0 \bmod 2$. Noting that this group contains $\Gamma(4)$, the principal congruence subgroup of level 4, we obtain the following theorem.

THEOREM II.3.2. *Let $\Lambda \subset Sp_n(\mathbb{Z})$ be a subgroup of $\Gamma(4)$. Then the restriction of the metaplectic extension to Λ is trivial. Moreover, there is a section $s: \Lambda \to \tilde{\Lambda}$ so that $\omega_\chi(s(\gamma)) = A(\gamma)$ for all $\gamma \in \Lambda$. In particular $\omega_\chi(s(\gamma))$ leaves Θ invariant for all $\gamma \in \Lambda$ and $\epsilon|s(\Lambda) \equiv 1$.*

REMARK. We have not proved the above theorem in the case $n = 1$. The theorem remains true since we may embed a two dimensional symplectic space (with the split quadratic form) in a larger one (the double for example). Then the case where $n = 1$ follows from the case of larger n together with the functoriality of ω_χ and A under restrictions.

For our applications of the oscillator representation to the theory of automorphic forms we will need to construct a certain line bundle over $\Lambda\backslash\mathfrak{H}_n$ which is the square-root of the locally homogeneous bundle associated to the determinant representation of the maximal compact subgroup U of $Sp(W)$. Let MU denote the 2-fold cover of U. We will call MU the meta-unitary group. There is a unique homomorphism $\sqrt{\det}$ making the following diagram commutative:

$$\begin{tikzcd} MU \arrow[r, "\sqrt{\det}"] \arrow[d] & S^1 \arrow[d, "\text{square}"] \\ U \arrow[r, "det"] & S^1 \end{tikzcd}$$

We recall that a manifold M of dimension n is said to have an almost complex structure if its tangent frame bundle can be reduced to a $U(n)$ principal bundle P. We will say that an almost complex manifold has a meta-unitary structure if there is an $MU(n)$ principal bundle $\bar{P}$ and a double covering $\bar{P} \to P$ which restricts to the cover $MU(n) \to U(n)$ on each fiber.

LEMMA 11.3.6. *$\Lambda \backslash \mathfrak{H}_n$ has a meta-unitary structue.*

PROOF: The tangent bundle of $\Lambda \backslash \mathfrak{H}_n$ may be reduced to the bundle:

$$U \to \Lambda \backslash Sp(W) \to \Lambda \backslash Sp(W)/U.$$

Recall we have a section $s: \Lambda \to \tilde{\Lambda}$ and $\Lambda' = s(\Lambda)$. Put $\bar{P} = \Lambda' \backslash \widetilde{Sp}(W)$. Since P is also equal to $\tilde{\Lambda} \backslash \widetilde{Sp}(W)$ we find that $\bar{P}$ is the required double cover of P.

We now define the line bundle L over $\Lambda \backslash \mathfrak{H}_n$ to be the line bundle associated to the principal bundle $\bar{P}$ and the representation $\sqrt{\det}$ of $MU(n)$. L is also the line bundle associated to the automorphy factor j on $Mp(W) \times \mathfrak{H}_n$ defined by $j(g, \tau) = \sqrt{\det} \, (c\tau + d)$ where $\sqrt{\ }$ is chosen so that $j(k, iI_n) = \sqrt{\det k}$ for $k \in MU(n)$. L is also a square root of determinant κ of the Hodge bundle over the space of principally polarized abelian varieties. We now make some remarks concerning the transformation law for the classical theta function.

There is a unique (up to scalar multiples) element φ_0 (the Gaussian) in the Schwartz space of E which satisfies:

$$\omega_\chi(k)\varphi_0 = \sqrt{\det\,(k)}^{-1}\,\varphi_0 \text{ for } k \in MU(n)$$

Then the function $\theta(g) = \Theta(\omega_\chi(g)\varphi_0)$ transforms according to:

(i) $\theta(\gamma g) = \epsilon(\gamma)\theta(g) \qquad \gamma \in \tilde{\Gamma}_Q$

(ii) $\theta(gk) = \sqrt{\det\,(k)}\sqrt{-1}\,\theta g \qquad k \in MU(n)$

Defining $\theta(\tau) = j(g_\tau, iI_n)\theta(g_\tau)$ where $g_\tau(iI_n) = \tau$, we find $\theta(\tau)$ satisfies for $\gamma \in \tilde{\Gamma}_Q$:

$$\theta(\gamma\tau) = \epsilon(\gamma)\sqrt{\det}(c\tau + d)\theta(\tau)$$

REMARKS. $\epsilon(\gamma)$ is called the *theta multiplier*. It is a character of $\tilde{\Gamma}_Q$. To deal with *compact* orthogonal locally symmetric spaces we need to prove the invariance of Θ under $\Lambda \subset Sp_n(\mathcal{O})$ for the integers $\mathcal{O}$ in a totally real field. Here Λ acts by a tensor product of oscillator representations. See Borel-Wallach [2], VIII.7.

4. **Dual reductive pairs and liftings**. Now let V be a real vector space of dimension

m equipped with a quadratic form (,) of signature (p, q). We assume there exists a lattice L in V so that (,) takes integral values on L. We let $Sp_n(\mathbb{R})$ denote the isometry group of the standard skew-symmetric form $\langle\ ,\ \rangle$ on $\mathbb{R}^{2n}$ and $Mp_n(\mathbb{R})$ denote the metaplectic group. Then $(\ ,\ ) \otimes \langle\ ,\ \rangle$ is a symplectic form on $W = V \otimes \mathbb{R}^{2n}$. We perform the construction of Section 1 to obtain the oscillator representation ω of $Mp(W)$. We let E denote the Lagrangian subspace of $\mathbb{R}^{2n}$ spanned by $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ the first n basis vectors in the standard basis for $\mathbb{R}^{2n}$. We then realize the Schrödinger model for this representation on $L^2(V \otimes E)$. We identify $V \otimes E$ with V^n, the n-fold direct sum of V with itself by means of the basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. We let F denote the subspace of $\mathbb{R}^{2n}$ spanned by the last n standard basis vectors. Then $\mathbb{R}^{2n} = E \oplus F$ is a Lagrangian splitting of $\mathbb{R}^{2n}$.

It is easily proved that the restriction of ω_χ to $Mp_n(\mathbb{R}) \subset Mp(W)$ is the m-fold tensor product f the oscillator representation of $Mp_n(\mathbb{R})$. We describe this representation on the elements $s(n(u))$ and $s(a(v))$ of section 2 (assuming det $v > 0$). Then the operators of ω on $f \in L^2(V^n)$ are given by:

(i) $\omega(s(n(u)))f(x) = e(1/2 \operatorname{tr} ({}^t xxu)f(x)$

(ii) $\omega(s(a(v)))f(x) = (\det v)^{m/2} f(xv)$

Here ${}^t xy$ is the $n \times n$ matrix of inner products (x_i, y_j). We have an embedding of $SO(p, q)$ into $Sp(W)$ which lifts to an embedding into $Mp(W)$ — by lemma II.2.1 (choose P to be the stabilizer of $V \otimes E$).

The operators $\omega_\chi(g)$ for $g \in SO(p, q)$ are given by $\omega_\chi(g) \cdot f(x) = f(g^{-1}x)$ for $f \in L^2(V^n)$. The operators $\omega_\chi(g')$ for $g' \in Mp_n(\mathbb{R})$ and $\omega_\chi(g)$ for $g \epsilon SO(p, q)$ commute with each other. Indeed $\tilde{O}(p, q)$ and $Mp_n(\mathbb{R})$ centralize each other in $Mp(W)$. In fact each is the full centralizer of the other. Such pairs of groups have been named dual reductive pairs by Roger Howe. There are many others besides the previous pair. In fact to treat the complex and quaternionic basic examples we would need the dual reductive pairs $U(n, n) \times U(p, q)$ and $SO^*(4n) \times Sp(p, q)$. In this paper we will restrict ourselves to the orthogonal case. We will rename ω to denote the representation of $Mp_n(\mathbb{R}) \times \tilde{O}(p, q)$ obtained by restricting ω_χ and will abbreviate $\omega(g', 1)$ and $\omega(1, g)$ to $\omega(g')$ and $\omega(g)$ for $g' \in Mp_n(\mathbb{R})$ and $g \in \tilde{O}(p, q)$.

There are two critical properties of dual reductive pairs which we now explain. The first involves a remarkable relation between the action of the invariant differential operators from $Sp_n(\mathbb{R})$ and $O(p, q)$. Let $\mathcal{U}(g)$ be the universal envelopping algebra of the Lie algebra g of $O(p, q)$ and $\mathcal{U}(g')$ be the universal envelopping algebra of the Lie algebra g' of Sp_n. Then we have the following theorem of Roger Howe [5].

THEOREM II.4.1. *Let* z *and* z' *be the centers of* $\mathcal{U}(g)$ *and* $\mathcal{U}(g')$. *Then* $d\omega(z)$ *and* $d\omega(z')$ *coincide as algebras of operators on the smooth vectors in* $L^2(V^n)$.

This theorem motivates the following lemma which may be proved by a direct calculation.

LEMMA II.4.1. *There is a scalar* λ *so that* $\omega(C') - \omega(C) = \lambda$ *where* C' *is the Casimir operator of* g' *and* C *is the Casimir operator of* g.

We have already met the second property — it is Theorem II.3.2. We intersect the subgroup $\Lambda' \subset \widetilde{Sp}(\mathbb{Z} \otimes L)$ with $Mp_n(\mathbb{R}) \times \tilde{O}(p,q)$ and we find a subgroup of finite index in the product of $\widetilde{Sp}_n(\mathbb{Z})$ and the integral points of $SO(p,q)$ which fixes Θ. We call this group $\Gamma' \times \Gamma$.

If $\varphi \in S(V^n)$ and $\Gamma' \times \Gamma$ is a suitable subgroup of the group of integral matrices in $Mp_n(\mathbb{R}) \times SO(p,q)$ then the function $\theta_\varphi(g', g)$ on $Mp_n(\mathbb{R}) \times \tilde{O}(p,q)$ given by:

$$\theta_\varphi(g', g) = \Theta(\omega(g', g)\varphi) = \sum_{x \in \mathbb{Z}^{mn}} \omega(g', g)\varphi(x)$$

satisfies

$$\theta_\varphi(\gamma' g', \gamma g) = \theta_\varphi(g', g).$$

Now suppose we can choose φ so that φ transforms according to a finite dimensional representation $\sigma' \otimes \sigma$ under $MU(n) \times (O(p) \times O(q))$. Then as described in Section 1, θ_φ can be identified with a section of the exterior tensor product $E_{\sigma'} \otimes E_\sigma$ which we denote $\theta_\varphi(x', x)$. In the cases we will consider $E_{\sigma'}$ and E_σ have natural metrics and θ_φ can be taken as the kernel of an integral operator mapping sections of $E_{\sigma'}$ to sections of E_σ and vice versa. We will use the notation of Howe, Piatetski-Shapiro [6] for this correspondence — if f is a section of $E_{\sigma'}$ then $\theta_\varphi(f)$ will be the section of E_σ obtained by taking the inner product of f and θ_φ. Similarly if η is a section of E_σ then $\theta_\varphi(\eta)$ will be the resulting section of $E_{\sigma'}$. In summary, we have:

$$\theta_\varphi(f) = (f, \theta_\varphi)$$

$$\theta_\varphi(\eta) = (\theta_\varphi, \eta)$$

where (,) are the Hilbert space inner products on section of $E_{\sigma'}$ and E_σ. We use the convention that the inner products are anti-linear in the first variable. $\theta_\varphi(\eta)$ will be redefined later.

REMARK: Again we should treat the general totally real case and again we refer to Borel-Wallach [2], VIII.7.

III. The construction of harmonic dual forms by a theta correspondence

1. **The Howe operator and the Schwartz form**. Our goal in this section is to construct a canonical element in $(\mathbf{A}^{qn}(D) \otimes \mathbf{S}(V \otimes X))^G$. Here X is a rational positive subspace of V, $\mathbf{A}^{qn}(D)$ denotes the space of smooth qn forms on D and the superscript G means the subspace of G invariants for the diagonal action. We think of an element F of the above space as a form on D with values in the Schwartz functions on $V \otimes X$. Given any linear functional T on $\mathbf{S}(V \otimes X)$ then $T(F(z))$ will be an ordinary differential form on D. If T is invariant under a discrete subgroup Γ of G then $T(F(z))$ will induce an element of $\mathbf{A}^{qn}(\Gamma \backslash D)$. In particular, if we choose $T = \Theta$ then we obtain a map:

$$(\mathbf{A}^{qn}(D) \otimes \mathbf{S}(V \otimes X))^G \to \mathbf{A}^{qn}(\Gamma \backslash D).$$

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For this reason it is important to construct forms of the above type.

We first construct an element of $(\mathbf{A}^0(D) \otimes \mathbf{S}(V \otimes X))^G$. We choose a majorant $(\ ,\ )_{z_0}$ for $(\ ,\ )$ as in I-1. Since $(\ ,\ )$ restricted to X is positive definite we obtain a positive definite form $((\ ,\ )) = (\ ,\ )_{z_0} \otimes (\ ,\ )$ on the tensor product. We choose an orthonormal basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ for X and identify X with R^n using this basis. Thus $V \otimes X$ is identified with V^n and we find that for $x, y \epsilon V^n$ the inner product $((x,y))$ corresponds to the trace of the matrix with i,j-th entry $(x_i, y_j)_{z_0}$. Since we have identified $\mathcal{S}(V \otimes X)$ with $\mathbf{S}(V^n)$ we have the operators of the oscillator representation of $Mp_n(R)$ acting on $\mathbf{A}^*(D) \otimes \mathbf{S}(V \otimes X)$ via the action on the second factor. This action commutes with the action of G on either factor, in particular it commutes with the diagonal action. We define the Gaussian $\varphi_0 \epsilon \mathbf{S}(V^n)$ by the formula:

$$\varphi_0(x) = \mathrm{e}^{-\pi((x,x))} = \prod_{i=1}^{n} \mathrm{e}^{-\pi(x_i,x_i)_{z_0}}.$$

Then φ_0 transforms by a character under K', the maximal compact subgroup $MU(n)$ of $Mp_n(\mathbb{R})$ which fixes J_0. This is easily proved by an infinitesimal computation using Lemma II.1.5. We define:

$$\bar{\varphi}_0(g,x) = \varphi_0(g^{-1}x).$$

Then $\bar{\varphi}_0 \epsilon \mathbf{A}^*(D) \mathbf{A}^0(D) \otimes \mathbf{S}(V^n))^G$.

We now look for a G invariant, K' semi-invariant, operator ∇ such that:

$$\nabla : (\mathbf{A}^*(D) \otimes S(V^n))^G \to (\mathbf{A}^{*+k}(D) \otimes S(V^n))^G$$

We give the construction of such an operator in the case $k = qn$ and $G = SO_0(p,q)$, the connected component of the identity of $O(p,q)$. In this case $K = SO(p) \times SO(q)$.

We first observe the isomorphism given by the restriction map:

$$(\mathbf{A}^*(D) \otimes S(V^n))^G \to (\Lambda^* T^*_{z_0}(D) \otimes S(V^n))^K.$$

But $T^*_{z_0}(D)$ is canonically isomorphic to $z_0^{\perp} \otimes z_0$. We obtain an isomorphism of the above space of G-invariants with $(\Lambda^*(z_0^{\perp} \otimes z_0) \otimes S(V^n))^K$. We are now in the framework studied by Howe [5]. We now write down an operator ∇' which is K invariant, K' semi-invariant and satisfies:

$$\nabla' : (\Lambda^*(z_0^{\perp} \otimes z_0) \otimes S(V^n))^K \to (\Lambda^{*+nq}(z_0^{\perp} \otimes z_0) \otimes S(V^n))^K.$$

Such an operator will give rise to the desired operator ∇.

We give a formula in coordinates for ∇'. We choose a basis $\{e_1, \dots, e_p, e_{p+1}, \dots, e_m\}$ compatible with the splitting $V = z_0^{\perp} \oplus z_0$. Then we let $\{x_{ij} : i = 1, 2, \dots, m;\ j = 1, 2, \dots, n\}$ denote coordinates relative the basis $\{e_i \otimes \epsilon_j\}$ for $\otimes$. We use the index convention that α, β will stand for indices between 1 and p and μ, ν for those between $p+1$ and m. We normalize the Riemannian metric on D to coincide on $T_{z_0}(D)$ with the negative of the tensor product $(\ ,\ ) \otimes (\ ,\ )$ restricted to $z_0^{\perp} \otimes z_0$. For this metric $\{e_\alpha \otimes e_\mu : 1 \leqq \alpha \leqq p,\ p+1 \leqq \mu \leqq p+q\}$ is on orthonormal basis. Using the metric, $e_\alpha \otimes e_\mu$ gives rise to an element $(e_\alpha \otimes e_\mu)^{\#}$ in $T^*_{z_0}(D)$ which we identify with the Maurer-Cartan form $\omega_{\alpha\mu}$ in p^*. This is a K-equivariant identification.

We have operators $\partial/\partial x_{ij}$, $M(x_{ij})$ on $S(V^n)$ where $M(x_{ij})$ denotes multiplication by x_{ij}. We also have operators $A(\omega_{ij})$ on $\Lambda^*(z_0^\perp \otimes z_0)$ where $A(\omega_{ij})$ denotes exterior multiplication by ω_{ij}. Then we define the Howe operator by:

$$\nabla' = \frac{(-1)^{nq}}{2^{nq/2}} \prod_{i=1}^{n} \prod_{\mu=p+1}^{m} \sum_{\alpha=1}^{p} \left(\frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha i}} - M(x_{\alpha i})\right) \otimes A(\omega_{\alpha\mu}).$$

Finally, we define:

$$\varphi = \nabla'\varphi_0 \in (\Lambda^{nq}(z_0^\perp \otimes z_0) \otimes S(V^n))^K.$$

In the next section we will need a formula for the element $\tilde{\varphi} \in (\mathbf{A}^{nq}(D) \otimes S(V^n))^G$ restricting to φ. We have identified $\Lambda^{nq}(z_0 \otimes z_0^\perp)$ with $\Lambda^{nq}p^*$. We look for a representation (for each x) of $\tilde{\varphi}$ as an element of $(\Lambda^{nq}p^* \otimes C^\infty(G))^K$. To do this we just use the map in the beginning of this section. We define

$$\tilde{\varphi}(g,x) \in \Lambda^{nq}p^* \otimes [C^\infty(G) \otimes S(V^n)]^G \text{ by } \tilde{\varphi}(g,x) = \varphi(g^{-1}x).$$

The K-invariance of φ_0 guarantees that for each x we have $\tilde{\varphi}(g,x) \in (\Lambda^{nq}p^* \otimes C^\infty(G))^K$ (we will sometimes write $\tilde{\varphi}(z,x)$ for $z \in D$ since $\tilde{\varphi}$ depends only on g mod K).

To make the previous correspondence completely clear we note that we may extend φ in terms of monomials ω_I in the $\omega_{\alpha\mu}$'s according to:

$$\varphi(x) = \sum_I f_I(x)\omega_I \qquad \text{(here } I \text{ is a multi-index)}$$

Then the f_I's satisfy:

$$f_I(kx) = \sum_J \sigma_{IJ}(k) f_J(x)$$

where σ_{IJ} is the IJ-th matrix element of K acting on $\Lambda^{nq}p^*$. We extend the ω_I's to left-invariant on G and we extend each $f_I(x)$ to $\tilde{f}_I(g,x) = f_I(g^{-1}x)$. Then

$$\tilde{\varphi}(g,x) = \sum_I \tilde{f}(g,x)\omega_I$$

We next note that for $n = 1$, the function $\varphi(x)$ is a linear combination of monomials of the form $\omega_{\alpha_1 p+1} \wedge \omega_{\alpha_2 p+2} \wedge \ldots \wedge \omega_{\alpha_q p+q}$ where $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ is an arbitrary subset of $\{1, 2, \ldots, p\}$. We will be especially interested in the coefficient $\eta_{I_0}(x)$ of $\omega_{1p+1} \wedge \ldots \wedge \omega_{1p+q}$. We see that this coefficient is given by:

$$\eta_{I_0}(x) = \frac{1}{2^{q/2}(2\pi)^{q/2}} H_q(\sqrt{2\pi}(e_1,x))\varphi_0(x)$$

where $H_q(t)$ is the q-th Hermite polynomial given by:

$$H_q(t) = (-1)^q e^{t^2} \frac{d^q}{dt^q} e^{-t^2}$$

We observe that the form $\bar{\varphi}$ for $n = 1$ determines the form $\bar{\varphi}$ for general n. Indeed, we have an isomorphism $\mathcal{S}(V)^{\otimes n}$ to $\mathcal{S}(V^n)$ sending $f_1 \otimes f_2 \otimes \ldots \otimes f_n$ to $\prod_{i=1}^{n} f_i$. We also have the n-th exterior power map. $\Lambda_q(z_0^\perp \otimes z_0) \to \Lambda^{nq}(z_0^\perp \otimes z_0)$. Clearly both of these maps are K-homomorphisms. Combining these two mappings we obtain a K-homomorphism (of degree n):

$$\Lambda^q(z_0^\perp \otimes z_0) \otimes S(V) \to \Lambda^{nq}(z_0^\perp \otimes z_0) \otimes S(V^n)$$

and consequently a map of K-invariants to be denoted $\wedge$:

$$\bigotimes_{1}^{n} (\Lambda^q(z_0^\perp \otimes z_0) \otimes S(V))^K \to (\Lambda^{nq}(z_0^\perp \otimes z_0) \otimes S(V^n))^K.$$

If $g \in G$ and $z = gz_0$ we let $(\ ,\ )_z$ denote the majorant of $(\ ,\ )$ associated to z. Then we have:

$$(x,y)_z = (g^{-1}x, g^{-1}y)_{z_0}.$$

We note the transformed Gaussian satisfies

$$\varphi_0(g^{-1}x) = \prod_{i=1}^{n} e^{-\pi(x_i,x_i)_z}.$$

We then have the following lemma whose proof is left to the reader.

LEMMA III.1.1.

$$\bar{\varphi}(z,x) = \bar{\varphi}_1(z,x_1) \wedge \bar{\varphi}_1(z,x_2) \wedge \ldots \wedge \bar{\varphi}_1(z,x_n).$$

NOTATION. $\bar{\varphi}_1(z,x_j)$ is the q-form obtaining by applying the partial Howe operator

$$\nabla_j = \frac{(-1)^q}{2^{q/2}} \prod_{\mu=p+1}^{m} \sum_{\alpha=1}^{p} \left(\frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha j}} - M(x_{\alpha j}) \right) \otimes A(\omega_{\alpha\mu})$$

to the Gaussian in the variable x_j.

We will later need a naturality property of the form $\bar{\varphi}_1(z,x)$ under restriction.

Let y be a vector in V of positive length and x another vector of positive length so that $x = x' + x''$ with x' a multiple of y and $(x'', y) = 0$. Let V_y denote the orthogonal complement of y in V, G_y be the subgroup of G which fixes y, D_y the set of negative q-planes contained in V_y *and* $i_y : D_y \to D$ the inclusion. We may consider the dual pair $Mp_n(\mathbb{R}) \times G_y \subset Mp(\mathbb{R}^{2n} \otimes V_y)$. The theory of the previous section produces an element $\bar{\varphi}' \in (\mathbf{A}^q(D_y) \otimes \mathbf{S}(V_y))^{G_y}$. We then have the following lemma.

LEMMA III.1.2.

$$i_y^* \bar{\varphi}(z,x) = \bar{\varphi}_0(x')\bar{\varphi}'(z,x'')$$

We now summarize the key tensorial properties of $\bar{\varphi}$ in the following theorem.

THEOREM III.1.1. (i) $\bar{\varphi}(z,x)$ *is a closed* nq*-form on* D *for every* x *in* V^n.
(ii) $\bar{\varphi}(z,x)$ *transforms under* $MU(n)$ *according to the representation* $(\sqrt{\det})^m$.

REMARK. We will not prove $\bar{\varphi}$ is closed. As the reader will observe in the section 3 this is essential. The proof may be found in Kudla-Millson [9].

2. **The theta correspondence**. In the last section we constructed a canonical element $\bar{\varphi} \in (\mathbf{G}^{nq}(D) \otimes \mathbf{S}(V^n))^G$. We now consider the element $\theta_\varphi \in \mathbf{A}^{nq}(\Gamma\backslash D) \otimes C^\infty(Mp_n(\mathbb{R}))$ defined by:

$$\theta_\varphi(g', z) = \Theta(\omega(g')\bar{\varphi}) = \sum_{x \in L^n}{}' \, \omega(g')\bar{\varphi}(z, x).$$

By $\sum'$, we mean the sum over only those $x \in L^n$ which are congruent to some fixed non-degenerate (i.e. rank n) n-tuple $x_0 \in L^n$ modulo some integer N. Elements of V^n will be called n-frames and elements of L^n will be called integral n-frames. We assume $\gamma \in \Gamma'$ implies $\gamma \equiv 1 \bmod N$.

Clearly, θ_φ defines a closed differential nq form on $M = \Gamma\backslash D$ for a suitable congruence subgroup (again denoted Γ) of the integral points of $O(p, q)$. The transformation law in g' is very subtle but is now clear (after the considerable work of Chapter II). Since Θ is invariant under Γ' we have

(i) $\theta_\varphi(\gamma' g', z) = \theta_\varphi(g', z)$

and since φ transforms under K' like $(\sqrt{\det})^m$ we have:

(ii) $\theta_\varphi(g'k', z') = [\sqrt{\det(k')}]^m \, \theta_\varphi(g', z)$.

The formulas (i) and (ii) together imply that θ_φ is a section of the line bundle L^m over $M' = \Gamma'\backslash\mathfrak{H}_n$ (recall $L = \kappa^{1/2}$). We use τ to denote the coordinate in $\S_n$. Then $\tau = u + iv$ with u and v real n by n symmetric matrices and v positive definite. We define an element $g'_\tau \in Mp_n(\mathbb{R})$, satisfying $g'_\tau(il_n) = \tau$, by the following formula:

$$g'_\tau = \begin{pmatrix} \sqrt{v} & \sqrt{v}^{-1}u \\ 0 & \sqrt{v}^{-1} \end{pmatrix}$$

Then we define $\theta_\varphi(\tau, z)$ by the formula:

$$\theta_\varphi(\tau, z) = j(g'_\tau, il_n)^{m/2}\theta_\varphi(g'_\tau, z) = (\det v)^{-m/4}\theta_\varphi(g'_\tau, z)$$

We may use θ_φ as a kernel of an integral transform as in Chapter II, section 4 and we obtain an integral transform:

$$\mathcal{L} : C_0^\infty(M', L^m) \to \mathbf{A}^{nq}(M).$$

Since θ_φ is closed we also obtain a map:

$$C_0^\infty(M', L) \to H^{nq}(M, \mathbb{R}).$$

Now L is a holomorphic line bundle. Holomorphic sections of L are classical Siegel modular forms; that is, holomorphic functions on $\mathfrak{H}_n$ satisfying the transformation law:

$$f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^{m/2} f(\tau) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \subset Sp_n(\mathbb{Z}).$$

We denote the holomorphic cusp-forms satisfying the above transformation law by

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$S_{m/2}(\Gamma')$. Clearly we can integrate a holomorphic cusp form against θ_φ and obtain a lifting $\mathcal{L}: S_{m/2}(\Gamma') \to \mathbf{A}^{nq}(M)$. A computation of Casimir values and Lemma II.4.1 yield the following theorem of Kudla-Millson [9].

THEOREM III.2.1. *The lift of a holomorphic cusp form is a closed harmonic nq form on M.*

We have constructed a mapping from spaces of classical Siegel modular forms to harmonic forms on locally symmetric spaces of orthogonal groups. We want to relate the image of this map to the dual classes of special cycles. Let $\mathbf{H}^{nq}$ denote the space of harmonic nq forms in M. We assume henceforth that M is compact.

We now define some linear combinations of special cycles. We will say an n-frame $x = \{x_1, x_2, \ldots, x_n\}$ has length equal to an n by n matrix β if β is equal to the matrix $((x_i, x_j))$. If $\beta > 0$ we write ${}^t xx = \beta$.

Let C'_β denote a set of Γ orbit representatives for the set of n-frames in L^n of length 2β saitsfying the previous congruence condition. Then $\mathcal{C}'_\beta$ is finite and we define:

$$C_\beta = \sum_{x \in \mathcal{C}'_\beta} C_x.$$

The notation C_x is explained in I.1.5.

We now define two subspaces of $\mathbf{H}^{nq}$. We let $\mathbf{H}$ denote the image of $S_{m/2}(\Gamma')$ under $\mathcal{L}$. We let $\mathbf{H}_{\text{cycle}}$ denote the span of the duals of the cycles C_β as above subject to the condition that β is positive definite. We will now outline a proof of the following theorem of Kudla-Millson. Let $m = p + q$.

MAIN THEOREM: *If $n < m/4$ then $\mathbf{H}_\theta = \mathbf{H}_{\text{cycle}}$.*

The theorem follows easily from a formula for certain Fourier coefficient of $\theta_\varphi(\eta)$ where η is a harmonic $(p - n)q$-form. We consider the non-singular pairing $[\ ,\ ]$ between harmonic $(p - n)q$ forms η and harmonic nq forms ω given by: $[\eta, \omega] = \int_M \eta \wedge \omega$. We define $\theta_\varphi(\eta)$ for η a harmonic $(p - n)q$ form by: $\theta_\varphi(\eta) = [\eta, \theta_\varphi]$. By the transformation law for θ we see that θ_φ is periodic with respect to the lattice $L' = \Gamma' \cap N$. Consequently $\theta_\varphi(\eta)$ has a Fourier expansion with respect to the characters of L'. Let $a_\beta(\theta_\varphi(\eta))$ denote the β^{th} Fourier coefficient for β an element of the dual lattice $(L')^*$ of L' (β will be a symmetric n by n matrix with rational entries). a_β is a function of v where $\tau = u + iv$. Then, for β positive definite, we have the following formula – to be proved in the next section:

$$a_\beta(\theta_\varphi(\eta))(v) = e^{-2\pi \operatorname{tr} \beta v} \int_{C_\beta} \eta \qquad (S)$$

We now show (S) implies the theorem. Clearly, it is enough to show $\mathbf{H}_\theta^\perp = \mathbf{H}_{\text{cycle}}^\perp$. This later equality we establish by proving two inclusions.

We first establish $\mathbf{H}_{\text{cycle}}^\perp \subset \mathbf{H}_\theta^\perp$. Accordingly, we assume η is orthogonal to the dual forms of the cycles C_β. Hence $\int_{C_\beta} \eta = 0$ for all cycles C_β with β positive definite and accordingly $a_\beta(\theta_\varphi(\eta)) = 0$ for all such β. But then any f in $S_{m/2}(\Gamma')$ has Fourier coefficients disjoint from $\theta_\varphi(\eta)$ and consequently we have

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$$[\theta_\varphi(f), \eta] = (f, \theta_\varphi(\eta)) = 0$$

We now establish $\mathbf{H}_\theta^\perp \subset \mathbf{H}_{\text{cycle}}^\perp$. We assume that $\theta_\varphi(\eta)$ is orthogonal to all holomorphic cusp forms. We introduce the Poincaré series (convergent provided $n < (p + q)/4$):

$$p_\beta(\tau) = c \sum_{\Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* e^{2\pi i \operatorname{tr} p\tau}}{j(\gamma, \tau)^m}$$

Here $\Gamma'_\infty = \Gamma' \cap N$ and c is a constant chosen so that $(p_\beta, f) = a_\beta(f)$ for $f \in S_{m/2}(\Gamma')$ and Γ'_∞ is the intersection of Γ' and N. We recall that $p_\beta(\tau)$ is a holomorphic cusp form.

We will also need the series:

$$p_\beta(\tau, s) = c(s) \sum_{\Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* e^{2\pi i \operatorname{tr} \beta\tau}}{j(\gamma, \tau)^m} \det v(\gamma\tau)^s$$

Here $c(s)$ is chosen so that $(p_\beta(\tau, s), f(\tau)) = a_\beta(f)$ for f a holomorphic cusp form. Then assuming $n < (p + q)/4$ we have $p_\beta(\tau, s)$ is holomorphic in s in a vertical half-plane containing 0 and $p_\beta(\tau, 0) = p_\beta(\tau)$.

Since $p_\beta(\tau)$ is a holomorphic cusp form we have:

$$(p_\beta(\tau, s), \theta_\varphi(\eta))\big|_{s=0} = (p_\beta(\tau), \theta_\varphi(\eta)) = 0$$

We now compute the first inner product directly. By the usual unfolding argument (valid for Re s sufficiently large) we obtain:

$$\begin{aligned}(p_\beta(\tau, s), \theta_\varphi(\eta)) &= c(s) \int_{\mathcal{F}_\infty} e^{-2\pi i \operatorname{tr} \beta\tau} (\det v)^{m/2+s} \theta_\varphi(\eta) \frac{du\,dv}{(\det v)^{n+1}} \\ &= c(s) \int_{\mathcal{P}_n} e^{-2\pi \operatorname{tr} \beta v} (\det v)^{m/2+s-n+1/2} a_\beta(\theta_\varphi(\eta)) \frac{dv}{(\det v)^{n+1/2}} \\ &= c(s) \left(\int_{C_\beta} {*\eta}\right) \int_{\mathcal{P}_n} e^{-4\pi \operatorname{tr} \beta v} (\det v)^{m/2+s-n+1/2} \frac{dv}{(\det v)^{n+1/2}}\end{aligned}$$

Here $\mathcal{F}_\infty$ is a fundamental domain for Γ'_∞ in $\mathfrak{H}_n$ and $\mathcal{P}_n$ is the space of positive definite symmetric n by n matrices.

The above integral formula coincides with $(p_\beta(\tau, s), \theta_\varphi(\eta))$ a priori only for Re s large, but, by the principle of unique analytic continuation, it must coincide with $(p_\beta(\tau, s), \theta_\varphi(\eta))$ in any region where they are both defined. The second integral has been computed in Siegel [18], Hilfsatz 37, and is convergent and non-zero provided Re $s > n - m/2$. This region includes zero under our assumption on n and m and consequently both the inner product and the integral are regular at $s = 0$. Evaluating the integral at $s = 0$ we obtain a non-zero constant and find:

$$cc' \int_{C_\beta} \eta = (p_\beta(\tau), \theta_\beta(\eta)) = 0$$

Hence the period of η over C_β is zero and the theorem is proved.

3. **The positive-definite Fourier coefficients of $\theta_\varphi(\eta)$.** The purpose of section 3 is to prove the formula:

$$a_\beta(\theta_\varphi(\eta))(v) = e^{-2\pi \operatorname{tr} \beta v} \int_{C_\beta} \eta$$

In this subsection we introduce some notation and some ideas concerning the cohomology of the total space of an oriented vector bundle.

If $x \in V^n$ then G_x will denote the stabilizer in G of the span of x and D_x will be the sub-symmetric space of D associated to G_x. We let $\Gamma_x = \Gamma \cap G_x$. We assume now that the span of x is a positive definite subspace of V.

The critical topological observation for what follows is that the space $E = \Gamma_x \backslash D$ is in a natural way a vector bundle over $C_x = \Gamma_x \backslash D_x$. There is a fibering $\pi: E \to C_x$ whose fibers are obtained by exponentiating the normal bundle of C_x in E. We choose some $x \in C'_\beta$ and study the cohomology of $E = \Gamma_x \backslash D$.

We review briefly some facts concerning the cohomology with compact supports of an oriented vector bundle over a compact manifold. These facts are proved in Chapter I, section 6 of Bott-Tu [3]. If E is an oriented vector bundle with fiber dimension nq then $H_c^{nq}(E, \mathbb{R})$ is isomorphic to $\mathbb{R}$ (here the subscript c denotes cohomology with compact support). Moreover, if φ is a closed, compactly-supported nq form then its image under the above isomorphism is the integral over any fiber (by Stokes theorem all fiber integrals are the same). If ψ is a closed compactly supported nq-form with period 1 along a fiber then ψ is called a Thom form and its class in $H_c^{nq}(E, \mathbb{R})$ is said to be the Thom class. It is a basic result in topology (proved in Bott-Tu) that the Thom class is Poincaré (or Lefschetz) dual to the zero section Z of E; that is, if η is any closed form on E we have:

$$\int_E \eta \wedge \psi = \int_Z \eta \qquad (D)$$

We now apply these considerations to $\pi: E \to C_x$ (so C_x is the zero section of our bundle). For any $g \in Sp_n(\mathbb{R})$ we find that $\omega(g')\bar{\varphi}$ is Γ_x invariant and hence we may project it to E. Let us abbreviate $\omega(g')\bar{\varphi}$ by $\bar{\varphi}$ until the end of Lemma III.3.4. This will cause no difficulty for the only additional properties of $\bar{\varphi}$ we will use are that $\bar{\varphi}$ is closed and rapidly decreasing (see below) both of which are true for all $\omega(g')\bar{\varphi}$. We would like to multiply $\bar{\varphi}$ by an appropriate factor so that it becomes a Thom form. Since $\bar{\varphi}$ is not compactly-supported we must make some modifications in the standard theory.

We note E inherits a Riemannian metric from D. We say a form τ on E is rapidly decreasing if we have for some constants C and any positive integer n: $\|\tau(z)\| \leq Ca(z)^{-n}$ where $a(z) = e^{r(z)}$ and $r(z)$ is the geodesic distance from the point $z \in D$ from the submanifold D_x. It is easy to see $\bar{\varphi}$ is rapidly decreasing. We recall also that $\bar{\varphi}$ is closed. The next lemma will allow us to replace $\bar{\varphi}$ by a compactly supported form.

LEMMA III.3.1. *$\bar{\varphi}$ is cohomologous to a compactly supported form via a rapidly decreasing primitive.*

PROOF. Let a_λ denote the operation of dilation by the positive number λ acting fiberwise on the vector bundle E. Then $r(\partial/\partial r)$ is the infinitesimal generator for the flow a_λ. Put $\tau = i_{r(\partial/\partial r)} \int_1^\infty a_\lambda^* \bar{\varphi}(d\lambda/\lambda)$. Then τ is defined in the complement of zero section of E and satisfies $d\tau = \bar{\varphi}$. We multiply τ by a smooth radial function σ vanishing at the zero section and identically 1 in the complement of a small neighbourhood fo the zero section. Then $\bar{\varphi}_c = \bar{\varphi} - d(\sigma\tau)$ is compactly supported. With this the lemma is proved.

We now treat the problem of whether (D) holds for $\bar{\varphi}$. Note that since $\bar{\varphi}$ is not compactly supported some restriction must be placed on η in order that the left-hand integral of (D) coverge. We allow η to be slowly increasing ($\|\eta\|$ bounded by a power of $a(z)$).

Let κ be the period of $\bar{\varphi}$ over any fiber of π and let η be any slowly increasing closed form on E. Then we have the following lemma.

LEMMA III.3.2.

$$\int_E \eta \wedge \bar{\varphi} = \kappa \int_{C_x} \eta$$

PROOF. We may write $\varphi = \psi + d\nu$ where ψ is compactly supported and ν is rapidly decreasing. Let $S(r)$ be the bundle of normal spheres of radius r around C_x in E. Then the volume of $S(r)$ is bounded by $a(z)^k$ for some fixed k (since the curvature of D is bounded below) and consequently:

$$\lim_{r\to\infty} \int_{S(r)\cap \text{fiber}} \nu = 0 \text{ and } \lim_{r\to\infty} \int_{S(r)} \eta \wedge \nu = 0$$

We conclude by Stokes Theorem that

$$\int_{\text{fiber}} \psi = \int_{\text{fiber}} \bar{\varphi} = \kappa \text{ and } \int_E \eta \wedge \bar{\varphi} = \int_E \eta \wedge \psi$$

Hence $1/\kappa\ \psi$ is a Thom form and

$$\int_E \eta \wedge \frac{1}{\kappa}\varphi = \int_E \eta \wedge \frac{1}{\kappa}\psi = \int_{C_x} \eta$$

Since κ is constant (in z) the lemma is proved.

REMARK. κ is a function of g' and x; that is, $\kappa = \kappa(g', x)$. Although it is initially defined only for rational x we may extend it to *all real* x with positive definite span by defining $\kappa(g', x)$ to be the integral of $\omega(g')\bar{\varphi}$ over some normal fiber of $D_x \subset D$.

Finally since $\bar{\varphi}$ is rapidly decreasing the sum $\Omega = 1/\kappa \sum_{\Gamma_x\backslash\Gamma} \gamma^* \bar{\varphi}$ converges, projects to M and defines the dual form to the cycle C_x in M by Lemma 2.1 of Kudla-Millson [8]. Rewriting slightly the condition that Ω is dual to C_x we obtain:

$$\int_M \eta \wedge \sum_{\Gamma_x\backslash\Gamma} \omega(g')\gamma^* \bar{\varphi} = \kappa(g', x) \int_{C_x} \eta$$

We may summarize the results obtained so far in this section by the following lemma.

LEMMA III.3.3. *Let φ be a rapidly decreasing closed nq form on $E = \Gamma_x\backslash D$ and η be a Γ-invariant form on E. Then the following are equivalent:*

(i) $\int_{\text{fiber}} \varphi = \kappa$

(ii) $\int_E \eta \wedge \varphi = \kappa \int_{C_x} \eta$

(iii) $\int_M \eta \wedge \sum_{\Gamma_x\backslash\Gamma} \gamma^* \varphi = \kappa \int_{C_x} \eta$

The previous lemma completes the topological preliminaries and we are now ready to study the integral for the β^{th} Fourier coefficient of $\theta_\varphi(\eta)$ namely:

$$a_\beta(\theta_\varphi(\eta))(v) = \frac{1}{\operatorname{vol} \mathcal{D}(v)} \int_{\mathcal{D}(v)} \theta_\varphi(\eta)(u + iv)e^{-2\pi i \operatorname{tr} \beta u}\, du.$$

Here $\mathcal{D}(v)$ is a fundamental domain for Γ'_∞ acting on the subset of $\mathfrak{H}_n$ defined by $\operatorname{Im} \tau = v$.

Let $\theta_\varphi(\tau, z, \beta)$ be the function defined by:

$$\theta_\varphi(\tau, z, \beta) = (\det v)^{-m/4} \sum_{x \in L^n}{}' \{\omega(g'_\tau)\bar{\varphi}(z, x) \colon (x, x) = 2\beta\}$$

Then an argument identical to that of Kudla-Millson [8], page 254, yields the following lemma.

LEMMA III.3.4.

$$a_\beta(\theta_\varphi(\eta))(v) = \int_M \eta \wedge \theta_\varphi(iv, z, \beta)$$

We note that $\theta_\varphi(\tau, z, \beta)$ is Γ invariant but is no longer Γ' invariant. We now rewrite $\theta_\varphi(\tau, z, \beta)$ as follows. Recall that we have chosen a set of representatives C'_β for the Γ-orbits of frames in L^n of length 2β. We define for $x \in C'_\beta$:

$$\theta_\varphi(\tau, z, x) = (\det v)^{-m/4} \sum_{\gamma \in \Gamma_x\backslash\Gamma} \omega(g'_\tau)\gamma^*\bar{\varphi}(z, x)$$

Recall in the previous section we defined:

$$\kappa(g', x) = \int_{\text{fiber}} \omega(g')\bar{\varphi}(z, x)$$

We define $\kappa'(\tau, x)$ by the formula:

$$\kappa'(\tau, x) = (\det v)^{-m/4}\, \kappa(g'_\tau, x)$$

We find then:

$$a_\beta(\theta_\varphi(\eta)(v) = \sum_{x \in C'_\beta} \int_M \eta \wedge (\det v)^{-m/4} \sum_{\Gamma_x\backslash\Gamma} \omega(g'_{iv})\gamma^*\bar{\varphi}(z, x)$$

hence, by Lemma III.3.3:

$$a_\beta(\theta_\varphi(\eta))(v) = \sum_{x\in C'_\beta} \kappa'(iv,x) \int_{C_x} \eta \qquad **$$

Thus, to prove (S) we must compute $\kappa'(iv,x)$. In fact it is equally easy to compute $\kappa'(iv,x)$, $\kappa'(\tau,x)$ or $\kappa(g',x)$. We note $\kappa'(iv,x) = (\det v)^{-m/4}\kappa(\sqrt{v},x)$.

The function $\kappa(g',x)$ has a large number of symmetries which we now describe. We first claim that $\kappa(g',x)$ depends (in the second variable) only upon $2\beta = (x,x)$. To see this it is sufficient to prove $\kappa(g',gx) = \kappa(g',x)$. But we may compute $\kappa(g',gx)$ by integrating $\omega(g')\bar{\varphi}(z,gx)$ along the transform by g of the fiber we used to calculate $\kappa(g',x)$ note $gz_0 \in D_{gx}$. But $\omega(g')\bar{\varphi}(z,gx) = \omega(g')\bar{\varphi}(g^{-1}z,x)$ and the result follows by the change of variable theorem.

Thus, there exists a smooth function κ'' on $G' \times \mathcal{P}_n$ with $\kappa(g',x) = \kappa''(g',\beta)$ for $\beta = {}^t xx$.

As a function of g', κ is determined by its restriction to $ML_n^+(\mathbb{R}) \subset Mp_n(\mathbb{R})$ since it transforms by characters under the actions of N and K'. Let u, v denote elements in $ML_n^+(\mathbb{R})$. Then we have:

$$\begin{aligned} \kappa(vu,x) &= (\det v)^{m/2}\kappa(u,xv) \\ \kappa''(vu,\beta) &= (\det v)^{m/2}\,\kappa''(u,{}^t v\beta v) \end{aligned} \qquad (A)$$

where v acts from the right on the frame x.

From the previous two paragraphs we see that it is sufficient to compute $\kappa(u,e)$ where $e = \{e_1, e_2, \ldots, e_n\}$ and $u \in ML_n^+(\mathbb{R})$. Let A' be the diagonal subgroup of $ML_n^+(\mathbb{R})$. We now claim that the function $\kappa(u,e)$ is determined by its restriction to A'.

LEMMA III.3.5. *$\kappa(u,e)$ is a spherical function on $ML_n^+(\mathbb{R})$.*

PROOF. By the Cartan decomposition for $ML_n^+(\mathbb{R})$we may write $u = k_1a'k_2$ with $k_1 \in s(SO(n))$, $k_2 \in s(SO(n))$ and $a' \in A'$. Clearly $\kappa(k_1a'k_2,e) = \kappa(k_1a',e)$ since $\bar{\varphi}$ is invariant under $s(SO(n))$. Also by (A) we have:

$$\kappa(k_1a',e) = \kappa(a',ek_1) = \kappa(a',e)$$

because $(ek_1,ek_1) = (e,e)$. With this the lemma is proved.

We now compute $\kappa'(\tau,x)$ in the case $n = 1$, this is the key calculation of the chapter. Let a'_μ be the diagonal matrix with diagonal entries (μ,μ^{-1}).

LEMMA III.3.6. *In case $n = 1$ we have:*

(i) $\kappa(a'_\mu,e_1) = \mu^{m/2}e^{-\pi\mu^2}$

(ii) $\kappa'(\tau,x) = e^{i\pi 2\beta\tau}$ *where* $(x,x) = 2\beta$

PROOF. We must compute the period of the form $\omega(g')\bar{\varphi}(z,x)$ along a fiber of the tube $E = \Gamma_x\backslash D$. We have reduced this to a study of $f(\mu) = \kappa(a'_\mu,e_1)$. In this case D_x is the set of negative q-planes contained in the orthogonal complement of e_1. We choose $z_0 \in D_x$ to be the q plane spanned by $e_{p+1}\wedge e_{p+2}\wedge\ldots\wedge e_{p+q}$. Then the fiber through z_0 is the set of negative q-planes pependicular to $\{e_2,e_3,\ldots,e_p\}$. This is a sub-symmetric space G_1/K_1 where G_1 is the subgroup of G which fixes $\{e_2, e_3, \ldots, e_p\}$. We find that p_1^* consists of the Maurer-Cartan forms $\omega_{1p+1}, \omega_{1p+2}, \ldots, \omega_{1p+q}$. Applying the formula I.2.4 we find that the period $f(\mu)$ of $\omega(a'_\mu)\varphi(g^{-1}e_1))$ is given by:

$$f(\mu) = \frac{1}{\operatorname{vol} K_1} \int_{G_1} \eta_{l_0}(a'_\mu, g^{-1}e_1)\, dg$$

where $\eta_{l_0}(a'_\mu, g^{-1}e_1)$ is the coefficient of $\omega_{1p+1} \wedge \omega_{1p+2} \wedge \ldots \wedge \omega_{1p+q}$ in the expression for $\omega(a'_\mu)\varphi(g^{-1}e_1)$ in terms of left-invariant q-forms on G. We have seen in the last section that:

$$\eta_{l_0}(a'_\mu, g^{-1}e_1) = \frac{1}{2^{q/2}(2\pi)^{q/2}} \mu^{m/2} H_q((\sqrt{2\pi}\, \mu g e_1, e_1)) e^{-\pi\mu^2\|g^{-1}e_1\|^2}.$$

Noting that the stabilizer of e_1 in G_1 is a maximal compact subgroup K_1 in G_1 we may compute the above integral by using the Cartan decomposition, $G_1 = K_1A_1K_1$. If a_r denotes the generic element of A_1 (with an upper 2 by 2 block of chr and shr) we obtain

$$\eta_{l_0}(a'_{\mu . a_r^{-1}} e_1) = \frac{1}{(2\pi)^{q/2}} \mu^{m/2} H_q(\sqrt{2\pi}\ \mu \operatorname{chr}) e^{-\pi\mu^2(\operatorname{ch}^2 r + \operatorname{sh}^2 r)}.$$

and:

$$f(\mu) = \frac{1}{2^{q/2}(2\pi)^{q/2}} \mu^{m/2} \operatorname{vol}(S^{q-1}) \int_0^\infty H_q(\sqrt{2\pi}\ \mu \operatorname{chr}) e^{-\pi\mu^2(\operatorname{ch}^2 r + \operatorname{sh}^2 r)} \operatorname{sh}^{q-1} r\, dr.$$

Hence

$$f(\mu) = \frac{1}{2^{q/2}(2\pi)^{q/2}} \frac{q\pi^{q/2}}{\Gamma((q/2)+1)} \mu^{m/2} e^{\pi\mu^2} \times \int_0^\infty H_q(\sqrt{2\pi}\ \mu \operatorname{chr}) e^{-2\pi\mu^2 \operatorname{ch}^2 r} \operatorname{sh}^{q-1} r\, dr.$$

We now compute

$$I(\mu) = \int_0^\infty H_q(\sqrt{2\pi}\ \mu \operatorname{chr}) e^{-2\pi\mu^2 \operatorname{ch}^2 r} \operatorname{sh}^{q-1} r\, dr$$

by first computing its Mellin transform $\mathcal{M}I(s)$. We have:

$$\begin{aligned} \mathcal{M}I(s) &= \int_0^\infty \int_0^\infty H_q(\sqrt{2\pi}\ \mu \operatorname{chr}) e^{-2\pi\mu^2 \operatorname{ch}^2 r} \operatorname{sh}^{q-1} r\, dr\, \mu^s \frac{d\mu}{\mu} \\ &= \int_0^\infty \int_0^\infty H_q(\sqrt{2\pi}\ \mu \operatorname{chr}) e^{-2\pi\mu^2 \operatorname{ch}^2 r} \mu^s \frac{d\mu}{\mu} \operatorname{sh}^{q-1} r\, dr \\ &= \frac{1}{(2\pi)^{s/2}} \int_0^\infty \int_0^\infty H_q(\mu) e^{-\mu^2} \mu^s \frac{d\mu}{\mu} \frac{\operatorname{sh}^{q-1} r}{(\operatorname{chr})^s}\, dr \\ &= \frac{1}{(2\pi)^{s/2}} \int_0^\infty H_q(\mu) e^{-\mu^2} \mu^s \frac{d\mu}{\mu} \int_0^\infty \frac{\operatorname{sh}^{q-1} r}{(\operatorname{chr})^s}\, dr. \end{aligned}$$

To obtain the first integral we have only to note:

$$\int_0^\infty H_q(\mu)e^{-\mu^2}\mu^s\,\frac{d\mu}{\mu} = (-1)^q\int_0^\infty \frac{d^q}{d\mu^q}\,e^{-\mu^2}\mu^s\,\frac{d\mu}{\mu}$$

$$= (s-1)(s-2)\ldots(s-q)\,\frac{1}{2}\,\Gamma\left(\frac{s-q}{2}\right).$$

As for the second integral, we leave to the reader the verification of the following formula (substitute $t = \mathrm{ch}^2 r$):

$$\int_0^\infty (\mathrm{ch}r)^a(\mathrm{sh}r)^b\,dr = \frac{\Gamma\left(-\frac{a+b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{2\Gamma\left(-\frac{a}{2}+\frac{1}{2}\right)}.$$

We then obtain (with $a = -s$, $b = q-1$):

$$\int_0^\infty \frac{(\mathrm{sh}r)^{q-1}}{(\mathrm{ch}r)^s}\,dr = \frac{\Gamma\left(\frac{s-q-1}{2}\right)\Gamma\left(\frac{q}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}$$

and accordingly (using the duplication formula for Γ):

$$\mathcal{M}I(s) = \frac{2^{q-2}\Gamma\left(\frac{q}{2}\right)}{(2\pi)^{s/2}}(s-1)(s-2)\ldots(s-q)\,\frac{\Gamma(s-q)}{\Gamma(s)}\,\Gamma\left(\frac{s}{2}\right)$$

and finally

$$\mathcal{M}I(s) = \frac{2^{q-2}}{(2\pi)^{s/2}}\,\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{s}{2}\right).$$

From this we obtain:

$$I(\mu) = 2^{q-1}\Gamma\left(\frac{q}{2}\right)e^{-2\pi\mu^2}$$

and

$$\kappa(a'_\mu, e_1) = f(\mu) = \mu^{m/2}e^{-\pi\mu^2}$$

Hence if $\lambda = \sqrt{2\beta}$ we obtain:

$$\kappa''(\mu, 2\beta) = \kappa(\mu, \lambda e_1) = \frac{1}{\lambda^{m/2}}\,\kappa(\mu\lambda, e_1) = \mu^{m/2}e^{-\pi\lambda^2\mu^2} = \mu^{m/2}e^{-\pi 2\beta\mu^2}$$

Now letting $\mu = \sqrt{v}$ we obtain:

$$\kappa'(iv, \lambda e_1) = \kappa(\mu, \lambda e_1) = \mu^{m/2}e^{-\pi 2\beta\mu^2} = v^{m/4}e^{-\pi 2\beta v}$$

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Using the transformation law under N and multiplying by $v^{-m/4}$ we obtain $\kappa'(\tau, x) = e^{i\pi 2\beta\tau}$ and the lemma is proved.

It is somewhat surprising that the formula for $\kappa(\tau, x)$ for general n follows from the formula for $n = 1$ by a simple formal argument. This is one reason for introducing the topological considerations of the beginning of this section.

Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be an n-tuple of positive real numbers and let a'_μ represent the matrix in A' with these diagonal entries. We have:

$$\omega(a'_\mu)\bar{\varphi}(z, e) = (\mu_1\mu_2 \dots \mu_n)^{m/2}\bar{\varphi}_1(z, \mu_1 e_1) \wedge \bar{\varphi}_1(z, \mu_2 e_2) \wedge \dots \wedge \bar{\varphi}_1(z, \mu_n e_n).$$

MAIN LEMMA.

$$\kappa(a'_\mu, e) = \kappa(a'_{\mu_1}, e_1)\kappa(a'_{\mu_2}, e_2) \dots \kappa(a'_{\mu_n}, e_n) = \prod_{i=1}^{n} \mu_i^{m/2} e^{-\pi\mu_i^2}$$

PROOF. By Lemma III.3.3 it is sufficient to prove that the form ψ_μ given by:

$$\psi_\mu = (\mu_1\mu_2 \dots \mu_n)^{-m/2} e^{\pi(\mu_1^2 + \mu_2^2 + \dots + \mu_n^2)}\omega(a'_\mu)\bar{\varphi}(z, e)$$

satisfies for any Γ-invariant closed form η on $\Gamma_e \backslash D_e$:

$$\int_{\Gamma_e \backslash D} \eta \wedge \psi_\mu = \int_{\Gamma_e \backslash D_e} \eta.$$

We define an $(n - 1)q$ form Φ_{n-1} on D by

$$\Phi_{n-1} = (\mu_1\mu_2 \dots \mu_{n-1})^{-m/2} e^{\pi(\mu_1^2 + \dots + \mu_{n-1}^2}\bar{\varphi}_1(z, \mu_1 e_1) \wedge \dots \wedge \bar{\varphi}_1(z, \mu_{n-1} e_{n-1})$$

We consider the integral:

$$I_n = \int_{\Gamma_e \backslash D} \eta \wedge \Phi_{n-1} \wedge \mu_n^{-m/2} e^{\pi\mu_n^2}\bar{\varphi}_1(z, \mu_n e_n).$$

We define a form Ω_{n-1} on $\Gamma_{e_n} \backslash D$ by:

$$\Omega_{n-1} = \sum_{\Gamma_e \backslash \Gamma_{e_n}} \gamma^* \Phi_{n-1}.$$

We claim $\|\Omega_{n-1}\|$ is a bounded function on $\Gamma_{e_n} \backslash D$. To see this we introduce one more function, a partial Gaussian F defined by:

$$F(g) = e^{-\pi/2(\|g^{-1}\mu_1 e_1\|^2 + \dots + \|g^{-1}\mu_{n-1} e_{n-1}\|^2)}.$$

Of course F depends only on g mod K. Our claim will be established if we prove that F satisfies the following three properties:

(1) $\|\Phi_{n-1}\| \leqq CF$ for some constant C

(2) $\sum_{\Gamma_e \backslash \Gamma_{e_n}} F(\gamma g)$ converges for $g \in G_{e_n}$

(3) $F(g_1 a_r k)$ is a non-increasing function of r where $g = g_1 a_r k$ is the Berger decomposition (see Rossman [17], page 169) with $g_1 \in G_{e_n}$, $k \in K$ and a_r as below.

Here a_r is the element of G defined by:

$$a_r e_i = e_i \text{ for } i \neq n \text{ or } i \neq p+1$$

$$a_r e_n = \operatorname{ch} r\, e_n + \operatorname{sh} r\, e_{p+1}$$

$$a_r e_{p+1} = \operatorname{sh} r\, e_n + \operatorname{ch} r\, e_{p+1}.$$

We leave (1) and (2) to the reader and prove (3). We note:

$$F(g_1 a_r k) = \mathrm{e}^{-(\pi/2)(\|a_r^{-1} g_1 \mu_1 e_1\|^2 + \ldots + \|a_r^{-1} g_1 \mu_{n-1} e_{n-1}\|^2)}$$

Property (3) then follows from the easy observation that if $x \epsilon V$ satisfies $(x, e_n) = 0$ then $\|a_r^{-1} x\|^2 \geqq \|x\|^2$. (In the previous formula $\| \;\|$ denoted the norm for the basic majorant $(\;,\;)_{z_0},)$.

Now we compute I_n by folding and unfolding the integral with respect to Γ_{e_n}.

$$\begin{aligned} I_n &= \int_{\Gamma_e \backslash D} \eta \wedge \Phi_{n-1} \wedge \mu_n^{-m/2} \mathrm{e}^{\pi \mu_n^2} \bar{\varphi}_1(z, \mu_n e_n) \\ &= \int_{\Gamma_{e_n} \backslash D} \eta \wedge \Omega_{n-1} \wedge \mu_n^{-m/2} \mathrm{e}^{\pi \mu_n^2} \bar{\varphi}_1(z, \mu_n e_n) \\ &= \int_{\Gamma_{e_n} \backslash D_{e_n}} i_n^*(\eta \wedge \Omega_{n-1}) \\ &= \int_{\Gamma_e \backslash D_{e_n}} i_n^*(\eta \wedge \Phi_{n-1}) \end{aligned}$$

The next to last inequality follows from the previous lemma and Lemma III.3.3. Here i_n denotes the inclusion of D_{e_n} into D or any quotient thereof.

We now apply Lemma III.1.2 which allows us to conclude that $i_n^* \bar{\varphi}(z, e_i) = \bar{\varphi}'(z, e_i)$ for $i = 1, 2, \ldots, n-1$. Thus the last integral above is in fact the integral I_{n-1} for the smaller orthogonal group $SO(p-1, q)$. Continuing in this way we obtain the lemma.

The formula (S) will follow from the next lemma.

LEMMA III.3.7.

$$\kappa'(\tau, x) = \mathrm{e}^{i\pi \operatorname{tr} 2\beta\tau} \textit{ where } (x, x) = 2\beta$$

PROOF. We have:

$$\kappa(a'_\mu, e) = (\mu_1 \mu_2 \ldots \mu_n)^{m/2} \mathrm{e}^{-\pi(\mu_1^2 + \mu_2^2 + \ldots + \mu_n^2)}$$

Hence, by Lemma III.3.5:

$$\kappa(u, e) = (\det u)^{m/2} \mathrm{e}^{-\pi \operatorname{tr} {}^t u u} \text{ for } u \in GL_n^+(\mathbb{R})$$

Let 2β be as above and write $2\beta = {}^t u_0 u_0$ for $u_0 \in GL_n^+(\mathbb{R})$. Then, by the formula (A), we have:

$$\begin{aligned}\kappa''(u, 2\beta) &= \kappa(u, eu_0) = (\det u_0)^{-m/2}\kappa(u_0u, e)\\ &= (\det u)^{m/2} e^{-\pi \operatorname{tr} {}^t u_0 u_0 {}^t u u}\\ &= (\det u)^{m/2} e^{-\pi \operatorname{tr} 2\beta {}^t u u}\end{aligned}$$

Letting $v = {}^t u u$ we obtain:

$$\kappa'(iv, eu_0) = \kappa(u, eu_0) = (\det u)^{m/2} e^{-\pi \operatorname{tr} 2\beta {}^t u u} = (\det v)^{m/4} e^{-\pi \operatorname{tr} 2\beta v}$$

Using the transformation law under N we obtain:

$$\kappa(g'_\tau, \beta) = (\det v)^{m/4} e^{i\pi \operatorname{tr} 2\beta\tau}$$

Hence:

$$\kappa'(\tau, x) = e^{i\pi \operatorname{tr} 2\beta\tau}$$

and

$$\kappa'(iv, x) = e^{-\pi \operatorname{tr} 2\beta v}$$

With this the lemma is proved. Substituting the value obtained for $\kappa'(iv, x)$ into ** we obtain the required formula for $a_\beta(\theta(\eta))$.

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ON THE EXISTENCE OF RESTRICTED K-LIMITS

BY

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ABSTRACT. The purpose of this paper is to generalize the Lindelöf-Čirka theorem.

1. **Introduction and notation**. Denote by B^n the unit ball in $\mathbb{C}^n$ and by $H(B^n)$ $(H^\infty(B^n))$ the (bounded) analytic functions on B^n.

A continuous curve $\Gamma: [0,1] \to \overline{B^n}$ is called *special* at $\xi \in \partial B^n$ if $\Gamma(t) \in B^n$, $t \in [0,1[\,; \Gamma(1) = \xi$ and if

$$\frac{|\Gamma(t) - \langle \Gamma(t), \xi\rangle \xi|^2}{1 - |\langle \Gamma(t), \xi\rangle|^2} \to 0, \qquad t \to 1.$$

For $A > 0$ we define an approach region

$$D_A(\xi) = \left\{ z \in B^n; |1 - \langle z, \xi\rangle| < \frac{A}{2}(1 - |z|^2) \right\}.$$

DEFINITION. A function f defined on B^n is said to have a K-limit at $\xi \in \partial B^n$ if $\lim_{z \to \xi} \{f(z): z \in D_A(\xi)\}$ exists for all A.

Koranyi [4] showed that if $f \in H^\infty(B^n)$ then f has K-limit for almost all $\xi \in \partial B^n$ (with respect to $d\sigma$, the Lebesgue measure on ∂B^n).

On the other hand, we have the following example:

EXAMPLE 1. (Čirka [2, p. 631]). Put $f(z,\omega) = \omega^2/(1 - z^2)$, then $f \in H^\infty(B^n)$ and $f(z,0) = 0$ so the radial limit of f at $\xi = (1,0)$ equals zero. But f has no K-limit at 1. Choose $c \in\,]0,1[$ and take $\Gamma(t) = (t, c\sqrt{1 - t^2})$. Then $f(\Gamma(t)) = c^2$ and since $|1 - \langle \Gamma(t), \xi\rangle| = |1 - t|$, $1 - |\Gamma(t)|^2 = 1 - t^2 - c^2(1 - t^2)$ we have

$$\frac{|1 - \langle \Gamma(t), \xi\rangle|}{1 - |\Gamma(t)|^2} = \frac{(1 - t)}{(1 - t^2)(1 - c^2)} = \frac{1}{(1 + t)(1 - c^2)}$$

so

$$\Gamma(t) \in D_{2/(1-c^2)}(1), \qquad \forall t \in [0,1[.$$

To avoid this we follow Čirka [2], and introduce restricted approach regions and limits.

Let $A > 0$ and $g(t)$ a positive decreasing function with

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$$\lim_{t \to 1} g(t) = 0 \qquad (*)$$

Put

$$R_{A,g}(\xi) = \left\{ z \in B^n; \frac{|1 - \langle z, \xi \rangle|}{1 - |z|^2} < \frac{A}{2}, \frac{|z - \langle z, \xi \rangle|^2}{1 - |\langle z, \xi \rangle|^2} < g(|z|) \right\}.$$

Each set $R_{A,g}(\xi)$ is then called a restricted approach region and if f is a function defined on B^n so that

$$\lim_{z \to \xi} \{ f(z) : z \in R_{A,g}(\xi) \}$$

exists for every restricted approach region $R_{A,g}(\xi)$, then f is said to have a restricted K-limit at ξ.

The Lindelöf-Čirka theorem [5], [2] now reads as follows.

THEOREM 1. *If $f \in H^\infty(B^n)$ and if $\lim_{t \to 1} f(\Gamma(t))$ exists for a special curve $\Gamma(t)$ with $\Gamma(1) = \xi$ then f has a restricted K-limit at ξ.*

(In particular, $\lim_{z \to \xi} \{ f(z) : z \in K \}$ exists where K is any cone in B^n with vertex at ξ).

We shall generalize this theorem for $f \in H(D_A(\xi))$ in the case where $\Gamma(t) = t\xi$, i.e., we shall assume that f has a radial limit at ξ.

Observe that, if $n \geq 2$ and if f has a restricted K-limit, then f has a *tangential* limit in certain directions. The curve

$$\Gamma(t) = (t, (1 - t^2)^{2/3}), \qquad t \in [0, 1]$$

shows this.

2. **The extremal function**. Let Ω be a bounded and open subset of $\mathbb{C}^n$ and put

$$F = \{ \varphi \in PSH(\Omega); \varphi \leq 0, \lim_{z' \to z} \varphi(z') \text{ exists } \forall z \in \partial\Omega \}.$$

If E is any subset of $\bar{\Omega}$ we define the extremal function h_E by

$$h_E(z) = \sup \{ \varphi(z); \varphi \in F, \varphi \leq -1 \text{ on } E \}, \qquad z \in \bar{\Omega}.$$

Denote by M_z, $z \in \bar{\Omega}$, the class of positive measures μ on $\bar{\Omega}$ such that

$$\varphi(z) \leq \int \varphi(\xi)\, d\mu(\xi), \forall \varphi \in F.$$

LEMMA 1. *For every compact set* K *in $\bar{\Omega}$ and every $z \in \bar{\Omega}$ there is a $\mu_z \in M_z$ so that*

$$-h_K(z) = \mu_z(K).$$

Proof. Proposition 2:1 in Cegrell [1].

LEMMA 2. *Assume that Ω is convex and that $\mu_\nu \in M_{z_\nu}$ where $z_\nu \to \xi$, $\nu \to +\infty$ and that ξ is a strictly convex boundary point. Then $\lim_{\nu \to +\infty} \mu_\nu(K) = 0$ for every compact subset of $\bar{\Omega}$ not containing ξ.*

Proof. The assumption implies that there is a convex function ψ on Ω, continuous up to the boundary such that $\psi(\xi) = 0$ but $\psi(z) < 0$ for $z \in \bar{\Omega} \setminus \{\xi\}$. Since convex functions are plurisubharmonic, the inequalities $\psi(z_\nu) \leq \int \psi(\xi)\, d\mu_\nu(\xi)$, $\nu \in N$ proves Lemma 2.

3. **The case $n = 1$.** If Ω is an open and bounded subset of the plane, then the extremal function h_E $(E \subset \bar{\Omega})$ is harmonic on $\Omega \setminus \bar{E}$.

PROPOSITION 1. *Let Ω be an open, bounded, convex and symmetric subset of the plane and let l denote the segment of symmetry. If E is the part of $\partial\Omega$ that lies on one given side of l then $h_E|_l = -\frac{1}{2}$.*

If K is any closed cone in Ω with vertex $\xi \in l \cap \partial\Omega$ such that $\mathrm{K} \setminus \{\xi\} \cap B(\xi, r) \subset \Omega$ for some $r > 0$ then there is an $\epsilon > 0$ so that

$$\mathrm{K} \cap B(\xi, \epsilon) \subset \{z \in \bar{\Omega};\ h_l(z) \leq -\epsilon\}.$$

($B(\xi, \epsilon)$ is the ball with center ξ and radius ϵ.)

Proof. The first part follows easily from the symmetry and the second part is proved by repeated use of symmetry.

PROPOSITION 2. *Let K be a closed convex cone with vertex ξ and l a line segment in K^0 with endpoint ξ. If $f \in H^\infty(\mathrm{K}^0 \cap B(\xi, r))$ for some $r > 0$ and if $\lim_{z \to \xi} \{f(z) : z \in l\}$ exists then $\lim_{z \to \xi} \{f(z) : z \in \mathrm{K}'\}$ exists for every closed cone K' in $\mathrm{K}^0 \cup \{\xi\}$ with vertex at ξ.*

Proof. (By a wellknown method). We can assume that $|f| \leq 1$ and that $\lim_{z \to \xi} \{f(z) : z \in l\} = 0$. Let K′ be given and choose $\epsilon > 0$ as in Proposition 1. For $z \in \mathrm{K}' \cap B(\xi, \epsilon)$ there is a $\mu_z \in M_z$ with $\mu_z(l) \geq \epsilon$. If we let z tend to ξ in K′ we get by Lemma 2 $\int \log |f(\eta)|\, d\mu_z(\eta) \to -\infty$, $z \to \xi$. But since $\log |f(z)| \leq \int \log |f(\eta)|\, d\mu_z(\eta)$ so $\lim_{z \to \xi} \{f(z) : z \in \mathrm{K}'\} = 0$ which completes the proof.

EXAMPLE 2. Proposition 2 cannot be generalized to bounded harmonic functions. Let E be the part of the unit circle in the lower halfplane. Then $h_E = -\frac{1}{2}$ on the real axes (by the observation in the beginning of this section). The restriction of h_E to segment of $y = x - 1$ in B^1 does not exceed $-\frac{3}{4}$. Thus h_E has a radial but not non-tangential limit.

As pointed out by J. C. Taylor, the next lemma is a consequence of Harnack's inequality.

LEMMA 3. *Let h be a positive harmonic function on $D_A(\xi) = \{z \in B^1;\ |\xi - z| < (A/2)(1 - |z|^2)\}$ $(\xi \in \partial B^1)$. If $\sup_{0<r<1} h(r\xi) < +\infty$ then $\sup_{z \in D_{A'}(\xi)} h(z) < +\infty$ for every $A' < A$.*

4. **Statement of the theorems**.

THEOREM 2. *Let $A > 0$ and g with property (*) be given. Assume that*

(1) $f \in H(D_A(\xi))$

(2) $\sup\left\{|f(\omega\xi)| : \omega \in B^1, \dfrac{|1 - \omega|}{1 - |\omega|^2} < \dfrac{A}{2}\right\} < +\infty$

(3) *$|f|$ has a plurisuperharmonic majorant ψ such that*

$$\lim_{\omega\to 1}\left\{\psi(\omega\xi)g(|\omega|)^{1/2} : \omega \in B^1, \frac{|1-\omega|}{1-|\omega|^2} < \frac{A}{2}\right\} = 0.$$

If $\lim_{r\to 1} f(r\xi)$ *exists, then* $\lim_{z\to\xi}\{f(z) : z \in R_{A',g}\}$ *exists for every* $A' < A$.

REMARK. Related results have been obtained by Cima and Krantz in [3]. However, our results also apply to non-normal functions, e.g.

$$z_2^2(1-z_1)^{-1}\log(1/1-z_1).$$

COROLLARY 1. *Assume that* $f \in H^1(B^n)$ *(i.e.,* $|f|$ *has a harmonic majorant). Then* $\lim_{r\to 1} f(r\xi) = f^*(\xi)$ *exists a.e.* $(d\sigma)$ *on* ∂B^n *and* $f^* \in L^1$ $(d\sigma)$. *If*

$$\sup_{0<r<1}\int P(r\xi,\eta)|f^*(\eta)|\,d\sigma(\eta) < +\infty$$

where

$$P(z,\eta) = \frac{(1-|z|^2)^n}{|(1-\langle z,\eta\rangle)|^{2n}}$$

and if $\lim_{r\to 1} f(r\xi)$ *exists, then* f *has a restricted K-limit at* ξ.

Proof. It follows from Koranyi [4] and Rudin [6, Theorem 5.4.12] that $|f|$ is bounded in every $D_A(\xi)$. Thus, Theorem 2 applies.

THEOREM 3. *Assume that*

(1) $f \in H(D_A(\xi))$

(2) $\sup\left\{|f(\omega\xi)| : \omega \in B^1, \frac{|1-\omega|}{1-|\omega|^2} < \frac{A}{2}\right\} < +\infty$

(3) $|f|$ *has a plurisuperharmonic majorant* ψ *such that*

$$\lim_{t\to 1}\{g(t+\delta(t-1))^{1/2}\psi(t\xi) : t \in \mathbb{R}\} = 0$$

for some $0 < \delta \leq 1$.

If $\lim_{r\to 1} f(r\xi)$ *exists then* $\lim_{z\to\xi}\{f(z) : z \in R_{A',g}\}$ *exists for every* $A' < A$.

COROLLORY 2. *Assume that* $f \in H(D_A(\xi))$ *and that* $|f|$ *has a plurisuperharmonic majorant* ψ *on* $D_A(\xi)$ *such that*

$$\sup_{0<r<1}\psi(\xi r) < +\infty.$$

If $\lim_{r\to 1} f(r\xi)$ *exists, then* $\lim_{z\to\xi}\{f(z) : z \in R_{A',g}\}$ *exists for every* $A' < A$.

Proof. Assumptions (1) and (3) in Theorem 3 are clearly fulfilled. It remains to prove that (2) holds. Denote by U the part of the complex line through zero and ξ that is contained in $D_A(\xi)$. The restriction of ψ to U is superharmonic (and not identically $+\infty$). Hence, there is a harmonic function h on U so that

$$|f(z)| \leq h(z) \leq \psi(z), \qquad \forall z \in U.$$

An application of Lemma 3 gives (2).

5. **Proof of Theorem 2.** Let $0 < A' < A$ and a sequence $z_\nu \in R_{A',g}$, $\lim_{\nu \to +\infty} z_\nu = \xi$ be given. To prove the theorem it is enough to prove that $\lim_{\nu \to +\infty} f(z_\nu) = \lim_{r \to 1} f(r\xi)$. Consider for $\lambda \in \mathbb{C}$, $(1 - \lambda)\langle z_\nu, \xi\rangle\xi + \lambda z_\nu$. This point is in $D_A(\xi)$ if and only if

$$|1 - \langle z_\nu, \xi\rangle| < \frac{A}{2}(1 - |(1 - \lambda)\langle z_\nu, \xi\rangle\xi + \lambda z_\nu|^2)$$

$$\Leftrightarrow |\langle z_\nu, \xi\rangle\xi + \lambda(z_\nu - \langle z_\nu, \xi\rangle\xi)|^2 < 1 - \frac{2}{A}|1 - \langle z_\nu, \xi\rangle|$$

$$\Leftrightarrow |\langle z_\nu, \xi\rangle|^2 + |\lambda|^2|z_\nu - \langle z_\nu, \xi\rangle\xi|^2 < 1 - \frac{2}{A}|1 - \langle z_\nu, \xi\rangle|$$

$$\Leftrightarrow |\lambda|^2 < \frac{-\frac{2}{A}|1 - \langle z_\nu, \xi\rangle| - |\langle z_\nu, \xi\rangle|^2 + 1}{|z_\nu - \langle z_\nu, \xi\rangle\xi|^2}.$$

But since $z_\nu \in R_{A',g}$ we have

$$|1 - \langle z_\nu, \xi\rangle| < \frac{A'}{2}(1 - |\langle z_\nu, \xi\rangle|^2)$$

so the right hand side above is not smaller than

$$\frac{-\frac{A'}{A}(1 - |\langle z_\nu, \xi\rangle|^2) - |\langle z_\nu, \xi\rangle|^2 + 1}{|z_\nu - \langle z_\nu, \xi\rangle\xi|^2} = \frac{\left(1 - \frac{A'}{A}\right)(1 - |\langle z_\nu, \xi\rangle|^2)}{|z_\nu - \langle z_\nu, \xi\rangle\xi|^2}$$

which in turn are not smaller than

$$\left(1 - \frac{A'}{A}\right) \cdot \frac{1}{g(|z_\nu|)}$$

again because $z_\nu \in R_{A',g}(\xi)$.

Hence, if

$$|\lambda|^2 \leq \left(1 - \frac{A'}{A}\right) \cdot \frac{1}{g(|z_\nu|)}$$

it follows that

$$\tau(\lambda) = (1 - \lambda)\langle z_\nu, \xi\rangle\xi + \lambda z_\nu \in D_A(\xi)$$

so $f(\tau(\lambda))$ is analytic in

$$|\lambda|^2 < \left(1 - \frac{A'}{A}\right) \cdot \frac{1}{g(|z_\nu|)}$$

and Cauchy's integral formula gives

$$f(\tau(1)) - f(\tau(0)) = \frac{1}{2\pi i} \int_{|\lambda| = [(1 - \frac{A'}{A}) \cdot \frac{1}{g(|z_\nu|)}]^{1/2}} \frac{f(\tau(\lambda))}{\lambda(\lambda - 1)} \, d\lambda.$$

Hence

$$(i)\qquad \begin{aligned}|f(z_\nu) - f(\langle z_\nu,\xi\rangle\xi)| &\le \frac{1}{2\pi}\int_{|\lambda|=[(1-\frac{A'}{A})\cdot\frac{1}{g(|z_\nu|)}]^{1/2}} \frac{|f(\tau(\lambda))|}{|\lambda||\lambda-1|}\\ &\le \frac{\psi(\langle z_\nu,\xi\rangle\xi)}{\left|\left[\left(1-\frac{A'}{A}\right)\frac{1}{g(|z_\nu|)}\right]^{1/2}-1\right|}.\end{aligned}$$

The right hand side of (i) equals

$$\frac{g(|z_\nu|)^{1/2}\psi(\langle z_\nu,\xi\rangle\xi)}{\left(1-\frac{A'}{A}\right)^{1/2}-g(|z_\nu|)^{1/2}} \to 0,\quad \nu\to+\infty.$$

by assumption 3. Thus,

$$f(z_\nu) - f(\langle z_\nu,\xi\rangle\xi) \to 0,\qquad \nu\to+\infty.$$

Now, by assumption 2, we can apply Proposition 2 and we have that $\lim_{\nu\to+\infty} f(\langle z_\nu,\xi\rangle\xi)$ exists and equals $\lim_{r\to1} f(r\xi)$. Thus, $\lim_{\nu\to+\infty} f(z_\nu) = \lim_{r\to1} f(r\xi)$ and the proof is complete.

6. **Proof of theorem 3**. Let $1 < A' < A$ be given and choose A'', $A' < A'' < A$ and let K_1 and K_2 be two cones with vertex at 1, symmetrical with respect to the real axis and so that

$$\left\{\omega\in\mathbb{C};\ |1-\omega| < \frac{A'}{2}(1-|\omega|^2)\right\} \subset K_1 \subsetneq K_2 \subset \left\{\omega\in\mathbb{C};\ |1-\omega| < \frac{A''}{2}(1-|\omega|^2)\right\}.$$

From now on, we think of $z\in B''$ to be close to ξ. For $z\in R_{A',g}$ denote by t_z the non-negative number such that $\langle z,\xi\rangle - t_z \perp \partial K_2$. Consider, for $\eta\in\mathbb{C}$,

$$L(\eta) = z + (\eta + t_z - \langle z,\xi\rangle)\xi.$$

There is a real number $K > 1$ (K independent of z) so that if $|\eta| \le K|\langle z,\xi\rangle - t_z|$ then $\eta + t_z \in K_2$ and a calculation shows that then $L(\eta)\in R_{A''',g}$ for every fixed A''', $A'' < A''' < A$.

Let now P be the Poisson kernel for some smooth simply connected domain $D \subset \{\eta\in\mathbb{C};\ |\eta| < K|\langle z,\xi\rangle - t_z|\}$ containing $\langle z,\xi\rangle - t_z$ and zero. Then

$$\begin{aligned}&|f(L(\langle z,\xi\rangle - t_z)) - f(\langle L(\langle z,\xi\rangle - t_z),\xi\rangle\xi)|\\ &\le \int_{\partial D} |f(L(\eta)) - f(\langle L(\eta),\xi\rangle\xi)|\,\mathrm{P}(\langle z,\xi\rangle - t_z,\eta)\,d\sigma(\eta)\\ &\le \text{(Harnack's inequality)}\\ &\le C\int_{\partial D} |f(L(\eta)) - f(\langle L(\eta),\xi\rangle\xi)\,\mathrm{P}(0,\eta)\,d\sigma(\eta) \le (i)\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{\partial D} \frac{\psi(\langle L(\eta),\xi\rangle\xi)\,\mathrm{P}(0,\eta)}{\left|\left(1-\frac{A'''}{A}\right)\frac{1}{g(|L(\eta)|)}\right|^{1/2}-1}\,d\sigma(\eta)\\
&\leq C\cdot C_1 \sup_{\partial D} g(|L(\eta)|)^{1/2}\int_{\partial D}\psi(\langle L(\eta),\xi\rangle\xi)\,\mathrm{P}(0,\eta)\,d\sigma(\eta)\\
&\leq C\cdot C_1 \sup_{\partial D} g(|L(\eta)|)^{1/2}\psi(\langle L(0),\xi\rangle\xi)\\
&= C\cdot C_1 \sup_{\partial D} g(|L(\eta)|)^{1/2}\psi(t_z\xi).
\end{aligned}
$$

The constant C (that comes from the Harnack inequality) depends on the shape of D, and not on the scale. We can thus take ∂D to be an ellipse with focus 0 and $\langle z,\xi\rangle - t_z$ so that

$$|\eta + t_z| \geqslant t_z + \delta(t_z - 1), \quad \forall \eta \in \partial D.$$

We find that

$$
\begin{aligned}
|f(z) - f(\langle z,\xi\rangle\xi)| &= |f(L(\langle z,\xi\rangle - t_z) - f(L(\langle\langle z,\xi\rangle - t_z),\xi\rangle)|\\
&\leq C\cdot g(t_z + \delta(t_z - 1))^{1/2}\psi(t_z\xi) \to 0, \qquad z \to \xi,
\end{aligned}
$$

and the proof is now finished in the same way as the end of the proof of Theorem 2.

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RESIDUAL CANONICAL CORRELATIONS

BY

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ABSTRACT. Residual canonical correlations are defined and are derived in terms of canonical correlations. Some measures of residual association are also defined, in terms of the residual canonical correlations and some possible applications are suggested.

1. **Introduction**. The sample canonical correlations $r_1, r_2, \ldots, r_p$ between two vector variables $\boldsymbol{x}$ and $\boldsymbol{y}$ ($\boldsymbol{x}$ of p components, $\boldsymbol{y}$ of q components, with $p \leq q$ without loss of generality) are measures of association between them, and are fairly well known statistics. In many situations, however, one needs to eliminate s ($s < p$) specified linear functions of $\boldsymbol{x}$ and obtain measures of the "residual" association between $\boldsymbol{x}$ and $\boldsymbol{y}$, that is left. These functions may either be known to the experimenter from prior information or might be under consideration for replacing the p x-variables by a smaller set. One needs these measures in Discriminant Analysis, Contingency Table Analysis, Econometrics and many other fields. The reason for eliminating s given functions may be that they are apriorily known to be irrelevant or it could also be that it is conjectured that these specified functions are the only ones that contribute to any meaningful association between $\boldsymbol{x}$ and $\boldsymbol{y}$. In the latter case then, the conjecture or hypothesis may be tested by the significance of the "residual" canonical correlations. In the former case, the residual canonical correlations are "refined" or "revised" measures when irrelevant functions are eliminated.

The idea of residual canonical correlations is originally due to Williams [5] (see also Kshirsagar [4]); but he introduced it only for the special case of $s = 1$. We consider this in a more general context and also develop some "overall" measures of residual association.

2. **Residual canonical correlations**. Let us denote by C_{xx}, C_{xy}, C_{yy} the matrices of the corrected sum of squares and products (S.S. & S.P.) of observations on $\boldsymbol{x}$ and $\boldsymbol{y}$ and based on n degrees of freedom (d.f.). This notation is due to Bartlett [1]. We shall further denote by $C_{xx\cdot y}$ the matrix

$$C_{xx} - C_{xy}C_{yy}^{-1}C_{yx}. \tag{2.1}$$

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which is the matrix of the "residual" S.S. & S.P. in the regression of $\boldsymbol{x}$ **on** $\boldsymbol{y}$. Let $\xi = L'x$, where L is $p \times s$, of rank s, denote the specified s linear functions of x, which are to be eliminated. Then we have the following Analysis of Dispersion table:

TABLE

Source	d.f.	S.S. & S.P. matrix
Regression of $\boldsymbol{y}$ on ξ	s	$C_{y\xi}C_{\xi\xi}^{-1}C_{\xi y}$
Regression of $\boldsymbol{y}$ on $\boldsymbol{x}$, given ξ	$p-s$	$C_{yx}C_{xx}^{-1}C_{xy} - C_{y\xi}C_{\xi\xi}^{-1}C_{\xi y} = C_{yy\cdot\xi} - C_{yy\cdot x}$
Residual in the regression of $\boldsymbol{y}$ on $\boldsymbol{x}$	$n-p$	$C_{yy\cdot x}$
Total	n	C_{yy}

The usual unrestricted Canonical Correlations $r_i (i = 1, \ldots, p)$ between $\boldsymbol{x}$ and $\boldsymbol{y}$ are obtained from the roots of the determinantal equation,

$$|-r^2C_{yy} + C_{yx}C_{xx}^{-1}C_{xy}| = 0, \tag{2.2}$$

or

$$|(1 - r^2)C_{yy} - C_{yy\cdot x}| = 0. \tag{2.3}$$

Correspondingly, we define the restricted canonical correlations $\phi_j (j = 1, \ldots, p - s)$ between $\boldsymbol{y}$ and $\boldsymbol{x}$, when $L'x$ is eliminated, to be obtainable from the equation.

$$|-\phi^2C_{yy\cdot\xi} + (C_{yy\cdot\xi} - C_{yy\cdot x})| = 0 \tag{2.4}$$

or

$$|(1 - \phi^2)C_{yy\cdot\xi} - C_{yy\cdot x}| = 0. \tag{2.5}$$

Williams [5], in the particular case $s = 1$, defined them alternatively as the principal semi-axes of the $(p - 1)$-dimensional ellipsoid obtained by the intersection of a p-dimensional ellipsoid and a hyperplane. However, in our case one would be dealing with a $(p - s)$-dimensional ellipsoid obtained by the intersection of a p-dimensional ellipsoid with a $(p - s)$ dimensional hyperplane.

To express the ϕ_j^2 in terms of the r_i^2 and ξ, we assume without loss of generality that the x's and y's are sample canonical variables, so that with the usual convention of scaling, we have

$$\begin{aligned} C_{xx} &= nI_p, \qquad C_{yy} = nI_q \\ \frac{1}{n}C_{xy} &= \left[\begin{array}{cccc|c} r_1 & & & & 0 \ldots 0 \\ & r_2 & & & \\ & & \ddots & & \\ & & & r_p & 0 \ldots 0 \end{array}\right] \\ &= [D\{r_i\}|0] \end{aligned} \tag{2.6}$$

where $D\{a_i\}$ will denote, in future, a diagonal matrix with diagonal elements $a_1, a_2, \dots, a_p$ and 0 will denote a null matrix of appropriate order. We also assume, without loss of generality, that the matrix L satisfies

$$L'L = I_s, \tag{2.7}$$

so that the s specified functions are orthonormal. Then

$$\frac{1}{n} C_{yy\cdot x} = I_q - \left[\begin{array}{c|c} D[r_i^2\} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{2.8}$$

and

$$\frac{1}{n} C_{yy\cdot \xi} = I_q - \left[\begin{array}{c|c} D\{r_i\}LL'D\{r_i\} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{2.9}$$

Therefore, (2.5) reduces to

$$|(1 - \phi^2)(I_p - D\{r_1\}LL'D\{r_i\}) - I_p - D\{r_i^2\}| = 0. \tag{2.10}$$

This can also be expressed as

$$|D\{\phi^2 - r_i^2\} + (1 - \phi^2)D\{r_i\}LL'D\{r_i\}| = 0 \tag{2.11}$$

Simplifying further, (2.10) or (2.11) becomes,

$$|D\{\phi^2 - r_i^2\}| \left| I_p + (1 - \phi^2) D\left\{\frac{r_i^2}{\phi^2 - r_i^2}\right\} LL'D\{r_i\} \right| = 0. \tag{2.12}$$

We now use,

$$|I_p + PQ| = |I_s + QP|, \tag{2.13}$$

where P, Q are matrices of order $p \times s$ and $s \times p$, respectively. This is

$$\left| I_s + (1 - \phi^2) L'D\left\{\frac{r_i^2}{\phi^2 - r_i^2}\right\} L \right| \prod_1^p (\phi^2 - r_i^2) = 0 \tag{2.14}$$

Using (2.7), this can also be written as

$$\left| L'D\left\{\frac{\phi^2(1 - r_i^2)}{\phi^2 - r_i^2}\right\} L' \right| \prod_1^p (\phi^2 - r_i^2) = 0. \tag{2.15}$$

The squared residual canonical correlations $\phi_j^2 (j = 1, \dots, p - s)$ are the roots of the equations (2.15).

3. **Measures of Residual Association**. In the statistical literature,

$$\sum_1^p r_i^2, \qquad \sum_1^p \frac{r_i^2}{1 - r_i^2}, \qquad \prod_1^p (1 - r_i^2)$$

are some of the important measures of association between $\boldsymbol{x}$ and $\boldsymbol{y}$. Actually, the last one is a measure of lack of association. We, therefore, propose

$$S_1 = \sum_{i}^{p} r_i^2 - \sum_{1}^{p-s} \phi_j^2 \tag{3.1}$$

$$S_2 = \sum \frac{r_i^2}{1 - r_i^2} - \sum \frac{\phi_j^2}{1 - \phi_j^2} \tag{3.2}$$

$$S_3 = \Pi(1 - r_i^2)/\Pi(1 - \phi_j^2) \tag{3.3}$$

as measures of overall residual association (or lack of association) between $\boldsymbol{x}$ and $\boldsymbol{y}$, when ξ is eliminated. We shall now express S_1, S_2, S_3 in terms of r_i^2 and $\boldsymbol{\xi}$.

From (2.10), we observe that $1 - \phi_j^2$ are the eigenvalues of the matrix

$$(I_p - D\{r_i\}LL'D\{r_i\})^{-1}(I_p - D\{r_i^2\}). \tag{3.4}$$

Hence, defining $\phi_j^2 = 0$ for $j > p - s$, and using tr for trace of a matrix,

$$\sum_{j=1}^{p} (1 - \phi_j^2) = \operatorname{tr}[(I_p - D\{r_i\}LL'D\{r_i\})^{-1}(I_p - D\{r_i^2\})] \tag{3.5}$$

Again, observing (see for example, Graybill [2])

$$(I_p - D\{r_i\}LL'D\{r_i\})^{-1} = I_p + D\{r_i\}L(I_s - L'D\{r_i^2\}L)^{-1}L'D\{r_i\} \tag{3.6}$$

Substituting (3.6) into (3.5), one gets

$$\begin{aligned} S_1 = \Sigma r_i^2 - \Sigma \phi_j^2 &= \operatorname{tr}[D\{r_i\}L(I_s - L'D\{r_i^2\}L)^{-1}L'D\{r_i\}(I_p - D\{r_i^2\})] \\ &= \operatorname{tr}[L'D\{r_i\}(I_p - D\{r_i^2\}D\{r_i\}L)(I_s - L'D\{r_i^2\}L)^{-1}] \\ &= \operatorname{tr}[L'D\{r_i^2\}(1 - r_i^2)\}L(L'D\{1 - r_i^2)\}L)^{-1}]. \end{aligned} \tag{3.7}$$

where tr AB = tr BA and $L'L = I_s$.

Similarly, from (3.4),

$$\prod_j (1 - \phi_j^2) = \frac{|I_p - D\{r_i^2\}|}{|I_p - D\{r_i\}LL'D\{r_i\}|}.$$

and, therefore, using (2.13) we get

$$S_3 = \frac{\Pi(1 - r_i^2)}{\Pi(1 - \phi_j^2)} = |L'D\{1 - r_i^2)\}L|. \tag{3.8}$$

To find S_2, we let

$$r_i^{*2} = \frac{r_i^2}{1 - r_i^2}, \qquad \phi_j^{*2} = \frac{\phi_j^2}{1 - \phi_j^2}. \tag{3.9}$$

Then from (2.10) or (2.11), we obtain, after some algebra,

$$|-\phi^{*2}I_p + D\{r_i^{*2}\} - D\{(1 + r_i^{*2})r_i\}LL'D\{r_i\}| = 0 \tag{3.10}$$

So, it follows that

$$\Sigma \phi_j^{*2} = \operatorname{tr}[D\{r_i^{*2}\} + D\{1 - r_i^{*2})r_i\}LL'D\{r_i\}], \tag{3.11}$$

or that (using tr AB = tr BA),

$$\begin{aligned} S_2 &= \Sigma r_i^{*2} - \Sigma \phi_j^{*2} \\ &= \operatorname{tr} [D\{(1 + r_i^{*2})r_i\}LL'D\{r_i\}] \\ &= \operatorname{tr} [L'D\{r_i\}D\{1 + r_i^{*2})r_i\}] \\ &= \operatorname{tr} [L'D\{r_i^2(1 + r_i^{*2})\}L] \\ &= \operatorname{tr} (L'D\{r_i^2/(1 - r_i^2)\}L). \end{aligned} \tag{3.12}$$

4. **Some applications**. If we consider a $(p + 1) \times (q + 1)$ contingency table with n_{ij} as the entry in the (i, j)-th call, and if we denote by x_i, y_j the dummy variables defined by

$$x_i = \begin{cases} 1, & \text{if a member belongs to the } i\text{-th row class,} \\ 0, & \text{otherwise } (i = 1, 2, \dots, p + 1) \end{cases}$$
$$y_j = \begin{cases} 1, & \text{if a member belongs to the } j\text{-th column class,} \\ 0, & \text{otherwise } (j = 1, 2, \dots, q + 1), \end{cases}$$

and carry out a canonical analysis between $\boldsymbol{x}$ and $\boldsymbol{y}$, it is well known that (see, for example Kendall & Stuart, [3])

$$n \sum_1^p r_i^2 = \chi^2 \tag{4.1}$$

Where χ^2 is the usual chi-square statistic for testing the independence of the row and column attributes, and n is the total frequency. Here $r_i^2 (i = 1, \dots, p)$ are the canonical correlations between $\boldsymbol{x}$ and $\boldsymbol{y}$, excluding the root 1, which is always present in such a case, since $\Sigma x_i = \Sigma y_j = 1$. If we quantify the row categories, according to some specified scores $a_1, a_2, \dots, a_{p+1}$ and wish to find out how much residual χ^2 is left out, for testing the goodness of these conjectured scores, we will have to eliminate two specified functions, one corresponding to the irrelevent scores $(1, 1, \dots, 1)$ leading to the irrelevant root, and the other corresponding to the hypothetical scores $(a_1, a_2, \dots, a_p)$ and use our above theory of residual canonical correlations to find ϕ_j^2 and employ

$$nS_1 = n(\Sigma r_i^2 - \Sigma \phi_j^2)$$

to test the significance of the residual association and if this is insignificant, the hypothesis of goodness of fit of the proposed scores is tenable. Bartlett [1] gives a numerical illustration of such a situation, where blood serological data are analysed and the goodness of fit of equidistant scores like $(1, 2, \dots, p)$ is tested.

In discriminant analysis with several groups, one can employ this technique to test the goodness of fit of 2 or more hypothetical discriminants. Williams [6] has given an example, where the relationship between certain physical variables about lamb carcasses and two factors, grade and weight is considered. In the response surface which expresses this relationship, he tests the hypothesis of the significance of only the main effects of these two factors. The main-effects can be represented by three specified functions and the hypothesis then can be tested by the significance or otherwise of a statistic like S_1.

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MORE ON CONVERGENCE OF CONTINUOUS FUNCTIONS AND TOPOLOGICAL CONVERGENCE OF SETS

BY

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ABSTRACT. Let $C(X, Y)$ denote the set of continuous functions from a metric space X to a metric space Y. Viewing elements of $C(X, Y)$ as closed subsets of $X \times Y$, we say $\{f_n\}$ converges topologically to f if $\mathrm{Li}\, f_n = \mathrm{Ls}\, f_n = f$. If X is connected, then topological convergence in $C(X, R)$ does not imply pointwise convergence, but if X is locally connected and Y is locally compact, then topological convergence in $C(X, Y)$ is equivalent to uniform convergence on compact subsets of X. Pathological aspects of topological convergence for seemingly nice spaces are also presented, along with a positive Baire category result.

1. **Introduction**. Let $\langle X, d_X\rangle$ be a metric space and let $\{C_n\}$ be a sequence of nonempty subsets of X. The *lower* and *upper closed limits* of $\{C_n\}$ are defined as follows [3]: $\mathrm{Li}\, C_n$ (resp. $\mathrm{Ls}\, C_n$) is the set of all points x each neighborhood of which meets all but finitely (resp. infinitely) many sets C_n. We say $\{C_n\}$ *converges topologically* to a (possibly empty) set C if $\mathrm{Li}\, C_n = \mathrm{Ls}\, C_n = C$. If $\langle Y, d_Y\rangle$ is another metric space, then we can regard members of $C(X, Y)$, the continuous functions from X to Y, as closed subsets of $X \times Y$. What does convergence of sequences in $C(X, Y)$ in the above sense mean? The relationship between topological convergence in $C(X, Y)$ and uniform convergence is explored in [2]. Here we consider in detail topological convergence versus pointwise convergence. In general both pointwise convergence and topological convergence in $C(X, Y)$ are weaker than Hausdorff metric convergence of graphs (induced by a metric compatible with the product uniformity) which is, in turn, weaker than uniform convergence. However, if $\{f_n\}$ converges to a uniformly continuous function f in the Hausdorff metric, then $\{f_n\}$ actually converges uniformly to f. In particular, if X is compact, then the Hausdorff metric on $C(X, Y)$ is topologically equivalent to the usual metric of uniform convergence [4]; this equivalence has been the basis for a number of papers in constructive approximation theory by B. Sendov, V. Popov, and their associates in Sofia (see, e.g., [5], [7] or [8]).

The relationship between topological convergence and pointwise convergence for general X and Y is a tenuous one. However, if X is locally connected and Y is locally compact, the situation can be described precisely: topological convergence in $C(X, Y)$ means uniform convergence on compact subsets of X.

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In the sequel (1) $S_\epsilon[x]$ will denote the open ball of radius ϵ about a point x in a metric space, (2) diam(A) will denote the diameter of a set A, (3)$\tilde{A}$ will denote the complement of A.

2. **Results**. We first resolve a simple question. Under what conditions on X and/or Y will pointwise convergence in $C(X, Y)$ force topological convergence? Essentially, X must be discrete.

THEOREM 1. *Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces. If X is discrete then pointwise convergence in $C(X, Y)$ implies topological convergence. Conversely, if $C([0, 1], Y)$ is nontrivial and pointwise convergence in $C(X, Y)$ implies topological convergence, the X is discrete.*

PROOF: Let X be a metric space without limit points. Suppose $\{f_n\} \subset C(X, Y)$ converges to a continuous function f pointwise. Immediately, we have $f \subset \operatorname{Li} f_n$. To show $\operatorname{Ls} f_n \subset f$, choose (x, y) off f, and let $\delta = \frac{1}{2}\min\{d_X(x, X - \{x\}), d_Y(y, f(x))\}$. Since $\lim_{n\to\infty} f_n(x) = f(x)$, eventually $d_Y(y, f_n(x)) > \delta$. Thus $S_\delta[x] \times S_\delta[y]$ is a neighborhood of (x, y) that meets at most finitely many members of $\{f_n\}$. We conclude $(x, y) \notin \operatorname{Ls} f_n$, whence $\operatorname{Li} f_n = \operatorname{Ls} f_n = f$.

Conversely suppose X is not discrete and $C([0, 1]), Y)$ is nontrivial, i.e., there exists $\varphi \in C([0, 1], Y)$ such that $\varphi(0) \neq \varphi(1)$. We construct a sequence $\{f_n\}$ in $C(X, Y)$ convergent pointwise to the function identically equal to $\varphi(0)$ on X that fails to converge topologically to f. Let x_0 be a limit point of X and let $\{x_n\}$ be a sequence in X convergent to x_0 such that for each n, $d_X(x_0, x_{n+1}) < d_X(x_0, x_n)$. Set $\alpha_n = d_X(x_0, x_n)$ and define $f_n \in C(X, Y)$ by

$$f_n(x) = \begin{cases} \varphi\left(\dfrac{1}{\alpha_n} d_X(x, x_0)\right) & \text{if } 0 \leq d_X(x, x_0) \leq \alpha_n \\ \varphi\left(2 - \dfrac{1}{\alpha_n} d_X(x, x_0)\right) & \text{if } \alpha_n < d_X(x, x_0) \leq 2\alpha_n \\ \varphi(0) & \text{if } d_X(x, x_0) > 2\alpha_n \end{cases}$$

Notice for each x in X eventually $f_n(x) = \varphi(0)$; so, $\{f_n\}$ converges to f pointwise. However, $\{(x_n, f_n(x_n))\}$ converges to $(x_0, \varphi(1))$, whence $\{f_n\}$ fails to converge topologically to f.

For general X and Y we can identify well-behaved sequences in $C(X, Y)$ for which pointwise convergence ensures topological convergence.

DEFINITION. $\Omega \subset C(X, Y)$ is called *pointwise equicontinuous* if for each $x \in X$ and each $\epsilon > 0$ there exists $\delta > 0$, perhaps dependent on x, such that whenever $f \in \Omega$ and $d_X(x, w) < \delta$ then $d_Y(f(x), f(w)) < \epsilon$.

THEOREM 2. *Let $\{f_n\}$ be a pointwise equicontinuous sequence in $C(X, Y)$ pointwise convergent to a continuous function f. Then $\{f_n\}$ converges topologically to f.*

PROOF: Since $f \subset \operatorname{Li} f_n$, if topological convergence does not occur, then we must have $\operatorname{Ls} f_n \not\subset f$. Pick $(x, y) \in \operatorname{Ls} f_n - f$ and choose $\epsilon < d_Y(y, f(x))$. By pointwise equicontinuity there exists $\delta > 0$ such that for each n, $d_X(w,x) < \delta$ implies $d_Y(f_n(w),f_n(x)) < \epsilon/3$. Choose $N \in Z^+$ so large that $d_Y(f_n(x),f(x)) < \epsilon/3$ whenever $n > N$. Since $(x,y) \in \text{Ls} f_n$, there exists $n > N$ and $w \in S_\delta[x]$ such that $d_Y(f_n(w), y) < \epsilon/3$. Together these facts yield $d_Y(f(x),y) < \epsilon$, a contradiction.

If X is locally connected and Y is locally compact, then the converse of Theorem 2 holds.

THEOREM 3. Let X be a locally connected metric space and Y a locally compact metric space. If $\{f_n\} \subset C(X, Y)$ converges topologically to a continuous function f, then $\{f_n\}$ converges pointwise to f and $\{f_n\}$ is pointwise equicontinuous.

PROOF: Suppose for some x, $\{f_n(x)\}$ fails to converge to $f(x)$. Then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for each k, $d_Y(f_{n_k}(x),f(x)) \geq \epsilon$. Also, since $\text{Li} f_n = f$ there exists a sequence $\{(w_k, f_{n_k}(w_k)\}$ convergent to $(x,f(x))$. Since X is locally connected, by passing to a subsequence we can assume that x and w_k lie in a common connected subset C_k of X where $\lim_{k\to\infty} \text{diam}(C_k) = 0$. Choose $\epsilon^* < \epsilon$ such that $E = \{y: d_Y(y,f(x)) = \epsilon^*\}$ is compact. For all k sufficiently large, $d_Y(f_{n_k}(w_k),f(x)) < \epsilon^*$ so that $f_{n_k}(C_k)$ meets both $\{y: d_Y(y,f(x)) < \epsilon^*\}$ and $\{y: d_Y(y,f(x)) > \epsilon^*\}$. Since each set $f_{n_k}(C_k)$ is connected, all but finitely many sets $f_{n_k}(C_k)$ meet E. Since E is compact, $\text{Ls}(f_{n_k}(C_k) \cap E)$ is nonempty. Choosing a point y_0 in this set, the condition $\lim_{k\to\infty} \text{diam}(C_k) = 0$ yields $(x, y_0) \in \text{Ls} f_n - f$, a contradiction.

Suppose now that $\{f_n\}$ is not equicontinuous at some x in X. Then there exists $\epsilon > 0$, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a sequence $\{z_k\}$ convergent to x such that for each k, $d_Y(f_{n_k}(z_k),f_{n_k}(x)) \geq \epsilon$. However, we know that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$; so, without loss of generality we can assume that for all k, $d_Y(f_{n_k}(z_k),f(x)) \geq \epsilon$. Arguing as in the first part of the proof with $(z_k,f_{n_k}(z_k))$ replacing $(x,f_{n_k}(x))$ for each k, we reach a contradiction in exactly the same manner.

Example 3 of [2] shows that the local compactness assumption for Y cannot be dropped, even if X is compact. Example 2 of [2] shows that the local connectedness assumption for X cannot be dropped, again, even if X is compact. However, this last example is not completely satisfying in that X is not connected. Now the main result of [2] says that if X is compact and connected and Y is locally compact, then topological convergence in $C(X, Y)$ ensures not only pointwise convergence but also uniform convergence. If X is merely connected and Y is locally compact, we cannot expect topological convergence to force uniform convergence (see Example 1 of [1]). But does it force pointwise convergence or pointwise equicontinuity? The answer is negative, even if $Y = R$.

EXAMPLE 1. In the plane for each $n \in Z^+$ let $E_n = \{(x,n + 1)\text{: } 1/(n + 1) \leq x \leq 1/n\}$; also define A and B as follows:

$$A = \{(0, y)\text{: } y \geq 0\}$$

$$B = \left\{(x, y)\text{: for some } n \in Z^+, x = \frac{1}{n} \text{ and } y \geq 0\right\}$$

Clearly $X = A \cup B \cup \bigcup_{n=1}^{\infty} E_n$ is a closed connected subset of the plane that fails to be locally connected. We define $f_n: X \to R$ by

$$f_n(x,y) = \begin{cases} 0 & \text{if } x > \dfrac{1}{n} \text{ or } y > n \\ n & \text{if } x \le \dfrac{1}{n} \text{ and } y < n - 1 \\ -ny + n^2 & \text{if } x \le \dfrac{1}{n} \text{ and } n - 1 \le y \le n \end{cases}$$

Notice that the graph of f_n restricted to each of the rays $x = 0$, $x = 1/n$, $x = 1/(n+1)$, $x = 1/(n+2), \dots$ consists of a horizontal segment at height n, a horizontal ray at height zero, and a segment joining them. The rest of the graph lies in the xy plane. It is easy to check that each such f_n is continuous. Now suppose f denotes the zero function. Since $f_n(0,0) = n$, the sequence $\{f_n\}$ does not converge pointwise to f. Also $\{f_n\}$ is not pointwise equicontinuous at the origin. However, we claim $\{f_n\}$ does converge topologically to f. First, it is obvious that $f \subset \operatorname{Li} f_n$ and that whenever $x > 0$ and $(x, y, z) \in \operatorname{Ls} f_n$, then $z = 0$. No problems can occur on the y-axis, either: if $y_0 \ge 0$ and $n - 2 \ge y_0$, then whenever $|y - y_0| < 1$ for all x we have either $f_n(x, y) = n$ or $f(x, y) = 0$. Thus, $(0, y_0, z) \in \operatorname{Ls} f_n$ implies $z = 0$. We have shown $\operatorname{Ls} f_n \subset f \subset \operatorname{Li} f_n$.

For arbitrary metric spaces X and Y if $\{f_n\}$ is a sequence in $C(X,Y)$ convergent pointwise to a continuous function f, then $\{f_n\}$ converges uniformly on compact subsets of X if and only if $\{f_n\}$ is pointwise equicontinuous. This observation, in conjunction with a standard diagonalization argument [6], is all there is to the following version of the Ascoli Theorem: Let X be a separable metric space, let Y be an arbitrary metric space, and let $\Omega \subset C(X,Y)$. Then each sequence $\{f_n\}$ in Ω has a subsequence convergent uniformly on compact subsets of X to some continuous function if and only if (i) for each x, $\Omega_x = \{g(x): x \in \Omega\}$ has compact closure in Y, (ii) Ω is pointwise equicontinuous. Theorems 2 and 3 now say that for arbitrary X and Y uniform convergence on compact subsets implies topological convergence, whereas if X is locally connected and Y is locally compact, then these notions of convergence in $C(X,Y)$ agree. The Ascoli Theorem translates as follows.

THEOREM 4. *Suppose X is a separable metric space and Y is an arbitrary metric space. Let $\Omega \subset C(X, Y)$, and consider the following statements.*

(1) Each sequence in Ω has a subsequence convergent topologically to a continuous function.

(2) Ω is pointwise equicontinuous and for each x, $\Omega_x = \{g(x): g \in \Omega\}$ has compact closure.

Condition (2) always implies condition (1), and if X is locally connected and Y is locally compact, then condition (1) implies condition (2).

Previous examples show that condition (1) need not imply either subcondition of (2) if X is connected and $Y = R$, or $X = [0, 1]$ and Y is a Hilbert space. However, separability of X is required to obtain condition (1) from condition (2).

EXAMPLE 2. If X is an arbitrary metric space, then by Zorn's Lemma there exists for each $\epsilon > 0$ a maximal subset A_ϵ of X such that whenever $\{w, z\} \subset A_\epsilon$ then $d_X(w, z) \geq \epsilon$. It follows that $X \subset \bigcup\{S_\epsilon[x]: x \in A_\epsilon\}$, so that if X is nonseparable some A_ϵ must be uncountable. Suppose such an A_ϵ has cardinal number at least c (which would be guaranteed for nonseparable X by the continuum hypothesis). Let $K = \{g: g: Z^+ \to \{0, 1\}$ and $g(n) = 1$ infinitely often$\}$, and let $\varphi: K \to A_\epsilon$ be an injection. For each g in K set $E(g) = \{n: g(n) = 1\}$. For each $n \in Z^+$ let $W_n \subset X$ be described as follows: $x \in W_n$ if there exists g_x in K such that $d_X(x, \varphi(g_x)) < \epsilon/3$ and n has an odd number of predecessors in $E(g_x)$. By the construction of A_ϵ the assignment $x \to g_x$ on W_n is well defined. We now define $f_n: X \to [0, 1]$ by the formula

$$f_n(x) = \begin{cases} \dfrac{3}{\epsilon} d_X(x, \varphi(g_x)) & \text{if } x \in W_n \\ 1 & \text{if } x \notin W_n \end{cases}$$

Note that each f_n is actually Lipschitz with Lipschitz constant $3/\epsilon$; so, condition (2) of Theorem 4 is satisfied. However, no subsequence of $\{f_n\}$ can converge topologically, because each subsequence is of the form $\{f_{h(n)}\}$ where h is an order isomorphism from Z^+ onto $E(g)$ for some $g \in K$, and by construction

$$\mathrm{Li}\, f_{h(n)} \cap (\{\varphi(g)\} \times [0, 1]\} = \emptyset$$

3. **Points of convergence of a topologically convergent sequence**. We first exhibit a sequence in $C(X, R)$ for a certain compact metric space X that is topologically convergent to a continuous function but which converges nowhere pointwise.

EXAMPLE 3. Let X denote the usual Cantor set in $[0, 1]$ and let $f: X \to R$ denote the zero function. Since X is a nowhere dense subset of $[0, 1]$ for each $j \in Z^+$ we can select j points $\{a_{j1}, a_{j2}, \ldots, a_{jj}\}$ in $[0, 1] - X$ satisfying

$$(1) \qquad a_{j1} < a_{j2} < \ldots < a_{jj}$$

$$(2) \qquad \bigcup_{i=1}^{j} S_{1/j}[a_{ji}] \supset [0, 1]$$

Set $I(j, 1) = [0, a_{j1}]$, $I(j, 2) = [a_{j1}, a_{j2}]$, $\ldots$, $I(j, j + 1) = [a_{jj}, 1]$ and let

$$\varphi: Z^+ \to \{(J, k): j \in Z^+, K \in Z^+ \text{ and } k \leq j + 1\}$$

lexicographically order the codomain. Denote $\varphi(n)$ by (j_n, k_n) and define $f_n: X \to R$ by

$$f_n(x) = \begin{cases} j_n & \text{if } x \in I(j_n, k_n) \\ 0 & \text{otherwise} \end{cases}$$

Since the endpoints of each interval $I(j_n, k_n)$ lie in $\{0, 1\} \cup ([0, 1] - X)$, each f_n is continuous. Since at each $x \in X$, $\{f_n(x)\}$ exceeds one frequently, $\{f_n\}$ converges nowhere pointwise to f. We claim, however, that $\mathrm{Li}\, f_n = \mathrm{Ls}\, f_n = f$. First, since lim diam

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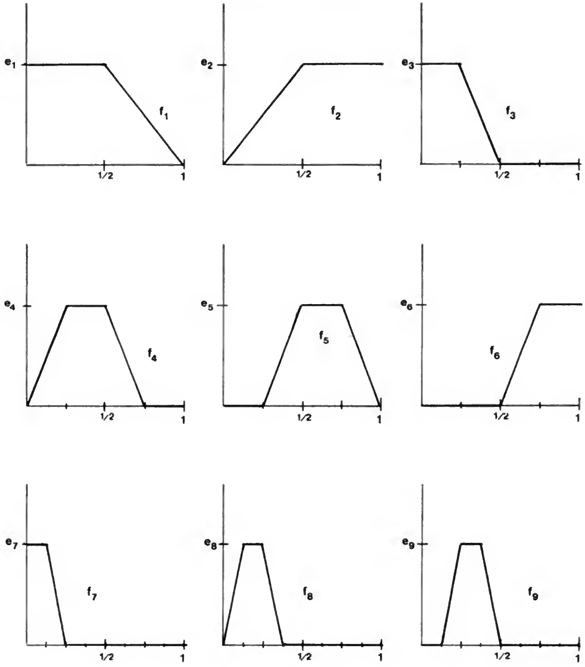

FIGURE 1.

$n \to \infty$

$(\{x\colon f_n(x) \neq 0\}) = 0$, we have $f \subset \mathrm{Li}\, f_n$. Also, for each $\alpha > 0$ the graphs of $\{f_n\}$ eventually all fail to meet $X \times (0, \alpha)$, and it follows that $\mathrm{Ls}\, f_n \subset f$. This establishes the topological convergence of $\{f_n\}$ to f.

We can also exhibit an example of nowhere pointwise convergence in the context of Example 3 of [2], i.e., $X = [0, 1]$ and $Y =$ Hilbert space of square summable sequences. Heeding the advice of Professor G. Piranian that "one filthy picture is worth a thousand dirty words", in lieu of an analytic argument, we sketch the graphs of the first nine terms of a sequence $\{f_n\}$ convergent topologically to the zero function but convergent nowhere pointwise (see Figure 1). As in Example 3 of [2], $\{e_i\colon i \in Z^+\}$ denotes the standard orthonormal basis in the Hilbert space.

We next present a Baire category result that says that if X is complete and Y is arbitrary and $\{f_n\}$ converges topologically to f, then $f(x)$ is a subsequential limit of $\{f_n(x)\}$ at most points.

THEOREM 5. *Let X be a complete metric space and let Y be any metric space. Let $\{f_n\}$ be a sequence in $C(X, Y)$ topologically convergent to f in $C(X, Y)$. Then there is a dense G_δ subset E of X such that for each x in E, $f(x)$ is a subsequential limit of $\{f_n(x)\}$.*

PROOF: For each $n \in Z^+$ and $\epsilon > 0$ we form the closed set

$$A(n, \epsilon) = \bigcap_{j=n}^{\infty} \{x\colon d_Y(f_j(x), f(x)) \geq \epsilon\}.$$

We claim that each such set is nowhere dense. Let $x_0 \in A(n, \epsilon)$ and $\lambda > 0$ be arbitrary. Choose $\lambda^* < \lambda$ for which $d_X(x, x_0) < \lambda^*$ implies $d_Y(f(x), f(x_0)) < \epsilon/2$. Since Ls $f_n \supset f$ we can find $j > n$ and $x \in X$ such that both $d_X(x, x_0) < \lambda^*$ and $d_Y(f(x_0), f_j(x)) < \epsilon/2$. It follows that $d_Y(f(x), f_j(x)) < \epsilon$, establishing the claim. Next for each pair of positive integers n and k let $B(n, k) =$ the complement of $A(n, 1/k)$, a dense open set. By the Baire Category Theorem

$$E = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} B(n, k)$$

is a dense G_δ set. If $x \in E$ then for each n and k let $j(n, k)$ be the smallest integer exceeding n for which $f_{j(n,k)}(x)$ has distance less than $1/k$ from $f(x)$. If we set $n_1 = j(1, 1)$ and for each $k > 1$ we let $n_k = j(n_{k-1}, k)$, we have $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

A second look at the proof of Theorem 5 reveals that we really did not need the full strength of $\mathrm{Li}\, f_n = \mathrm{Ls}\, f_n = f$, but only $f \subset \mathrm{Ls}\, f_n$. With this weaker assumption, our result has a rather nice interpretation: if we can approach each (x, y) in f along some trajectory of the form $\{(x_k, f_{n_k}(x_k))\}$, then we can approach most points along a vertical trajectory. Under the stronger assumption $\mathrm{Li}\, f_n = \mathrm{Ls}\, f_n = f$, must there actually exist a subsequence of $\{f_n\}$ convergent to f on some dense G_δ subset of X? The answer is negative, even if X is compact.

EXAMPLE 4. Let μ denote Lebesgue measure on the line, and let X be a Cantor set of positive measure in [0, 1] as constructed in [6]. Note that for each x_0 in X and each $\epsilon > 0$, $\mu(\{x\colon x \in X \text{ and } |x - x_0| < \epsilon\}) > 0$. For each $n \in Z^+$ let V_n be the union of a finite collection of disjoint open intervals $\{W_{ni}\colon i = 1, 2, 3, \ldots, k_n\}$ in (0, 1) such that

(1) For each n and $i \leq k_n$ the endpoints of W_{ni} lie in $[0, 1] - X$.
(2) For each n and $i \leq k_n$, $W_{ni} \cap X \neq \emptyset$.
(3) For each $x \in X$ and $n \in Z^+$ there exists $y \in V_n$ such that $|x - y| < 1/n$.
(4) For each n, $\mu(V_n) < 1/n$.

By condition (1) each set $V_n \cap X$ is clopen in X. Define for each n, $f_n\colon X \to R$ by

$$f_n(x) = \begin{cases} n & \text{if } x \notin V_n \\ 0 & \text{if } x \in V_n \end{cases}$$

Since $V_n \cap X$ is clopen in X, each f_n is continuous, and conditions (2), (3) and (4) imply that $\{f_n\}$ converges topologically to the zero function. Let $\{f_{n_k}\}$ be an arbitrary subsequence of $\{f_n\}$. For each k write A_k for $V_{n_k} \cap X$, denote the closed set $\bigcap_{n=k}^{\infty} A_n$ by B_k. Clearly,

$$\{x\colon \lim_{k\to\infty} f_{n_k}(x) = 0\} = \bigcup_{k=1}^{\infty} B_k.$$

Now whenever $n \geq k$ we have $B_k \subset A_n$; so, condition (4) implies that for all k, $\mu(B_k) = 0$. Since $\operatorname{int}_X(B_k) \neq \emptyset$ would imply $\mu(B_k) > 0$, we conclude that each B_k is nowhere dense in X. Thus $\{x\colon \lim_{k\to\infty} f_{n_k}(x) = 0\}$ is a set of first category in X, and since X is complete $\{x\colon \lim_{k\to\infty} f_{n_k}(x) = 0\}$ contains no dense G_δ set.

We remark in closing that a somewhat simpler counterexample can be constructed for $X = [0, 1]$ and $Y =$ the Hilbert space of square summable sequences. We leave this to the imagination of the reader.

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TWO INEQUALITIES FOR PLANAR CONVEX SETS

BY

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ABSTRACT. B. Grünbaum, J. N. Lillington and lately R. J. Gardner, S. Kwapien and D. P. Laurie have considered inequalities defined by three concurrent straight lines in the interior of a planar compact convex set. In this note we prove two elegant conjectures by R. J. Gardner, S. Kwapien and D. P. Laurie.

1. **Introduction**. Trying to establish a conjecture of B. Grünbaum [2], J. N. Lillington [3] came up with some interesting problems concerning the division of a planar compact convex set by three concurrent lines. In [1] R. J. Gardner, S. Kwapien and D. P. Laurie solve a conjecture of J. N. Lillington [3] and propose the following two new conjectures concerning area inequalities for planar convex sets.

Let X be a planar compact convex set and L_1, L_2, L_3 three concurrent lines through the interior point 0, which divide X into six regions with areas $|X_i|$, $|Y_i|$, $i = 1, 2, 3$ (see figure 1).

(Here and throughout we denote by $|E|$ the area of the set E. Values of i lying outside the set $\{1, 2, 3\}$ are defined by $i \equiv i + 3$).

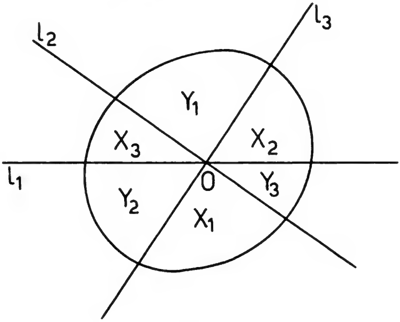

FIG. 1.

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We define

$$Q(X) = \frac{|X_1|}{|Y_1|} + \frac{|X_2|}{|Y_2|} + \frac{|X_3|}{|Y_3|},$$

$$P(X) = \frac{|X_1| + |X_2|}{|Y_3|} + \frac{|X_2| + |X_3|}{|Y_1|} + \frac{|X_3| + |X_1|}{|Y_2|}$$

R. J. Gardner, S. Kwapien and D. P. Laurie conjectured:

1. $Q(X) \geq 3/2$,
2. $P(X) \geq 3$.

2. **Proof of the first conjecture**. We originally obtained a trigonometrical proof for the first conjecture using an affine transformation and high school mathematics only. Below we give another proof, following R. J. Gardner's, S. Kwapien's and D. P. Laurie's formulation.

THEOREM 1. $Q(X) \geq 3/2$.

PROOF. The line L_i intersects X at the points p_i, q_i, $i = 1, 2, 3$. Let $A_1 = q_1p_2 \cap p_1q_3$, $A_2 = q_2p_3 \cap p_2q_1$, $A_3 = q_3p_1 \cap p_3q_2$.

CASE 1. *Suppose O is an interior point of the triangle $A_1A_2A_3$ (see figure 2). Obviously we have*

$$Q(X) \geq Q(A_1A_2A_3). \tag{1}$$

We use areal coordinates, setting

$$A_1 = (1, 0, 0), \qquad A_2 = (0, 1, 0), \qquad A_3 = (0, 0, 1)$$

$O = (\chi_1, \chi_2, \chi_3)$, where $\chi_1 + \chi_2 + \chi_3 = |A_1A_2A_3| = 1$, $\chi_i \geq 0$. We take L_i to be the line

$$\chi'_{i+1} - \chi_{i+1} = \lambda_i(\chi_{i-1} - \chi'_{i-1}), \qquad \lambda_i > 0$$

and

$$p_i = (p^i_1, p^i_2, p^i_3), \qquad q_i = (q^i_1, q^i_2, q^i_3).$$

An easy calculation gives:

$$p^i_{i-1} = \chi_{i-1} + \chi_{i+1}/\lambda_i, \quad p^i_i = \chi_i + \left(1 - \frac{1}{\lambda_i}\right)\chi_{i+1}, \quad p^i_{i+1} = 0,$$

$$q^i_{i-1} = 0, \quad q^i_i = \chi_i + (1 - \lambda_i)\chi_{i-1}, \quad q^i_{i+1} = \chi_{i+1} + \lambda_i\chi_{i-1},$$

and

$$\begin{aligned} &|Op_{i-1}q_{i+1}| = \chi_i^2\left(\lambda_{i+1} + \left(\frac{1}{\lambda_{i-1}}\right) - 1\right) \\ &|A_ip_{i+1}Oq_{i-1}| = (\chi_{i-1} + \chi_{i+1})^2 - \lambda_{i-1}\chi_{i+1}^2 - \chi_{i-1}^2/\lambda_{i+1}, \qquad i = 1, 2, 3. \end{aligned} \tag{2}$$

These expressions were obtained in [1].

Now using the inequality

$$k_1a_1^2 + k_2a_2^2 \geq \frac{k_1k_2(a_1 + a_2)^2}{k_1 + k_2}, \tag{3}$$

where $k_1, k_2, a_1, a_2 \in R$, $k_1 + k_2 > 0$, we have

$$\lambda_{i-1}\chi_{i+1}^2 + \frac{\chi_{i-1}^2}{\lambda_{i+1}} \geq \frac{\dfrac{\lambda_{i-1}}{\lambda_{i+1}}(\chi_{i+1} + \chi_{i-1})^2}{\lambda_{i-1} + \dfrac{1}{\lambda_{i+1}}}. \tag{3a}$$

Consequently, from (2) and (3a) we obtain:

$$\frac{|Op_{i-1}q_{i+1}|}{|A_ip_{i+1}Oq_{i-1}|} \geq \left[\frac{\chi_i}{\chi_{i-1} + \chi_{i+1}}\right]^2\left[\lambda_{i+1} + \frac{1}{\lambda_{i-1}}\right],$$

or,

$$Q(A_1A_2A_3) \geq \sum_{i=1}^{3}\left[\frac{\chi_i}{\chi_{i-1} + \chi_{i+1}}\right]^2\left[\lambda_{i+1} + \frac{1}{\lambda_{i-1}}\right]$$

or,

$$Q(A_1A_2A_3) \geq \sum_{i=1}^{3}\left(\left[\frac{\chi_{i-1}}{\chi_i + \chi_{i+1}}\right]^2\lambda_i + \left[\frac{\chi_{i+1}}{\chi_i + \chi_{i-1}}\right]^2\frac{1}{\lambda_i}\right),$$

We have now

$$\left[\frac{\chi_{i-1}}{\chi_i + \chi_{i+1}}\right]^2\lambda_i + \left[\frac{\chi_{i+1}}{\chi_i + \chi_{i-1}}\right]^2\frac{1}{\lambda_i} \geq \frac{2\chi_{i-1}\chi_{i+1}}{(\chi_i + \chi_{i+1})(\chi_{i-1} + \chi_i)} \tag{4}$$

and consequently

$$Q(A_1A_2A_3) \geq \sum_{i=1}^{3}\frac{2\chi_{i-1}\chi_{i+1}}{(\chi_i + \chi_{i+1})(\chi_{i-1} + \chi_i)}.$$

We use now the known inequality

$$\sum_{i=1}^{3}\frac{2\chi_{i-1}\chi_{i+1}}{(\chi_i + \chi_{i+1})(\chi_i + \chi_{i-1})} \geq \frac{3}{2} \tag{5}$$

and finally obtain $Q(X) \geq 3/2$.

The equality holds if and only if X is a triangle and L_1, L_2, L_3 are parallel straight lines through the centroid to the sides respectively. This can be seen by making use of (1), (3a), (4) and (5).

Case 2. *Suppose that O is not an interior point of the triangle $A_1A_2A_3$ and that O lies in the angle A_1 of the triangle $A_1A_2A_3$. We can prove*

$$\frac{|X_2|}{|Y_2|}+\frac{|X_3|}{|Y_2|}>\frac{3}{2}.$$

PROOF. The straight lines through p_1 parallel to the line q_1p_2 intersects L_2, L_3 and A_2A_3 at the points N, M, A_3' respectively. The proof of theorem 1 in case 1 is independent of the angle A_1 of the triangle $A_1A_2A_3$. Consequently we can consider as a triangle the figure S with sides A_2A_3', the semiline A_2p_2 and the semiline $A_3'p_1$. Then, from case 1, it follows that $Q(S)>3/2$ (this can be also proved directly).

Also, it is very easy to see that:

$$\frac{|X_1|}{|Y_1|}+\frac{|X_2|}{|Y_2|}+\frac{|X_3|}{|Y_3|}=Q(X)>Q(S).$$

So, for the second case, we have $Q(X)>3/2$.

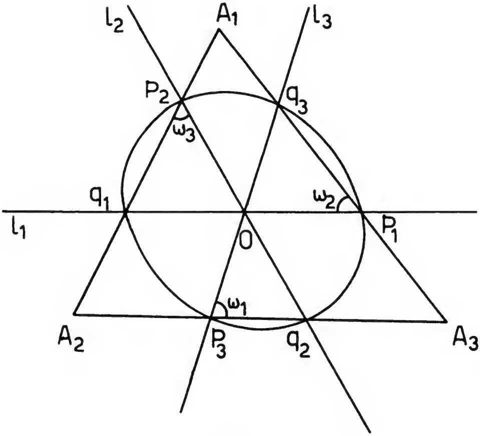

FIG. 2.

3. **Proof of the second conjecture.**

THEOREM 2. $P(X)\geq 3$.

We similarly have to investigate two cases.

CASE 1. Suppose that the point O is an interior point of the triangle $A_1A_2A_3$ (see figure 2). From figure 2, we can see that

$$P(X) \geq P(A_1A_2A_3).$$

We need the following lemma.

Lemma. *Let ABC be a triangle and O, N, M points on the sides BC, CA, AB, respectively. We will show that:*

$$P_A = \frac{|BOM| + |OCN|}{|AMON|} \geq \frac{\sin (B + \omega) \sin \phi}{\sin (\omega + \phi) \sin B} + \frac{\sin (C + \phi) \sin \omega}{\sin (\omega + \phi) \sin C} - 1$$

where A, B, C are the angles of the triangle ABC and $\measuredangle BOM = \omega \measuredangle CON = \phi$.

The equality holds if and only if MN is parallel to BC.

Proof of the lemma. It is elementary to see that:

$$\frac{|BOM|}{|ABC|} = \frac{\overline{BM} \cdot \overline{BO}}{c \cdot a} = \frac{d_1 \cdot \overline{BO}^2}{c \cdot a},$$

$$\frac{|OCN|}{|ABC|} = \frac{\overline{CN} \cdot \overline{CO}}{b \cdot a} = \frac{d_2 \cdot \overline{CO}^2}{b \cdot a},$$

where

$$d_1 = \frac{\sin \omega}{\sin (B + \omega)}, \qquad d_2 = \frac{\sin \phi}{\sin (C + \phi)}$$

and a, b, c are the sides of the triangle ABC. Using the key inequality (3) we obtain

$$\frac{|BOM| + |OCN|}{|ABC|} \geq \frac{a \cdot d_1 \cdot d_2}{bd_1 + cd_2}.$$

Consequently

$$P_A = \frac{1}{\dfrac{|ABC|}{|BOM| + |OCN|} - 1} \geq \frac{1}{\dfrac{bd_1 + cd_2}{ad_1d_2} - 1}. \tag{6}$$

Using simple trigonometrical formulas in the triangle ABC we obtain:

$$\frac{bd_1 + cd_2}{ad_1d_2} = \frac{\sin B \sin (C + \phi)}{\sin A \sin \phi} + \frac{\sin C \sin (B + \omega)}{\sin A \sin \omega} \tag{7}$$

$$= \frac{\sin B \sin C}{\sin A} [\cot \phi + \cot \omega] + 1.$$

From (6) and (7) it follows that

$$P_A \geq \frac{\sin A}{\sin B \sin C [\cot \phi + \cot \omega]}. \tag{8}$$

It is very easy to see that the following identity holds

$$\text{(9)}\qquad \frac{\sin A}{\sin B \sin C[\cot \phi + \cot \omega]} = \frac{\sin (B + \omega) \sin \phi}{\sin B \sin (\phi + \omega)} + \frac{\sin (C + \phi) \sin \omega}{\sin C \sin (\phi + \omega)} - 1 .$$

Formulas (8) and (9) prove our lemma.

The equality in (3) holds when $k_1\alpha_1 = k_2\alpha_2$ or,

$$\frac{d_1 \cdot \overline{BO}}{c} = \frac{d_2 \cdot \overline{CO}}{b},$$

or,

$$\frac{\overline{BM}}{c} = \frac{\overline{CN}}{b},$$

that is, MN is parallel to BC.

We are now ready to prove the second conjecture. In figure 2 we define:

$$\measuredangle Op_3q_2 = \omega_1, \qquad \measuredangle Op_2q_1 = \omega_3, \qquad \measuredangle Op_1q_3 = \omega_2 .$$

Applying the lemma to the triangles $q_1A_1p_1$, $q_2A_2p_2$, $q_3A_3p_3$, we take,

$$\begin{aligned} P_{A_i} &= \frac{|q_iOp_{i+1}| + |Op_iq_{i-1}|}{|A_ip_{i+1}Oq_{i-1}|} \\ &\geq \frac{\sin \omega_{i-1} \sin (\pi - A_{i-1} - \omega_i + \omega_{i+1})}{\sin (\pi - A_i - \omega_{i+1}) \sin (\pi - A_{i+1} - \omega_{i-1} + \omega_i)} \\ &\qquad + \frac{\sin (\pi - A_1 - \omega_{i+1} + \omega_{i-1}) \sin (\pi - A_{i-1} - \omega_i)}{\sin \omega_{i+1} \sin (\pi - A_{i+1} - \omega_{i-1} + \omega_i)} - 1 \end{aligned}$$

Now

$$P(A_1A_2A_3) = \sum_{i=1}^{3} P_{A_i},$$

or

$$\text{(10)}\qquad P(A_1A_2A_3) \geq \sum_{i=1}^{3} \left(\frac{m_{i-1}}{n_{i+1}} + \frac{n_{i+1}}{m_{i-1}}\right) - 3,$$

where,

$$m_{i-1} = \sin \omega_{i-1} \sin (\pi - A_{i-1} - \omega_i + \omega_{i+1})$$

$$n_{i+1} = \sin (\pi - A_i - \omega_{i+1}) \sin (\pi - A_{i+1} - \omega_{i-1} + \omega_i) .$$

Obviously it follows

$$P(A_1A_2A_3) \geq 2 + 2 + 2 - 3 = 3 .$$

The equality holds, according to our lemma, if p_2q_3 is parallel to L_1, p_3q_1 is parallel to L_2 and p_1q_2 is parallel to L_3. Also taking (10) and (11) into account we find that L_1, L_2, L_3 must be parallel to the sides of $A_1A_2A_3$. Therefore we conclude that the equality holds if and only if X is a triangle, O is its centroid and L_1, L_2, L_3 are parallel to the sides, respectively.

CASE 2. *A similar argument holds as in Theorem 1, case 2.*

4. **Comments**. The theorems 1 and 2 are remarkable tools in proving inequalities on convex sets. R. J. Gardner, S. Kwapien and D. P. Laurie noticed (see [1] page 309) that their theorems 3.1 and 4.1 follow immediately from theorems 1 and 2 respectively. Also it is worthwhile to notice here that Grünbaum's inequality $f(X) \geq 1/2$ (see [2]) follows easily from theorem 1.

The author is grateful to the referee for several improvements in the paper.

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23 PLATONOS STR.
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EXTENDED CENTROIDS OF SKEW POLYNOMIAL RINGS

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ABSTRACT. Let R be a prime ring with $\sigma \in \mathrm{Aut}(R)$. We determine the extended centroid of the skew polynomial ring $R[x, \sigma]$ when (*i*) $\langle\sigma\rangle$ is X-outer of finite order, (*ii*) $\langle\sigma\rangle$ is X-outer and infinite, (*iii*) σ^m is X-inner and no smaller power of σ fixes the extended centroid of R.

Suppose $R \subseteq S$ are prime rings with extended centroids C and D respectively. A natural question to consider is the relationship between C and D. In this paper we investigate the situation where R is a prime ring and S is the skew polynomial ring over R with respect to $\sigma \in \mathrm{Aut}(R)$, i.e., $S = R[x, \sigma]$. We use the notions of X-inner and X-outer automorphisms due to Kharchenko. In sections 2 and 3, the following cases are considered: (*i*) $\langle\sigma\rangle$ is X-outer of finite order m; (*ii*) $\langle\sigma\rangle$ is X-outer and infinite; (*iii*) m is the least positive integer such that σ^m is X-inner and no smaller power of σ fixes C.

We prove the extended centroid of $R[x, \sigma]$ is isomorphic to: (*i*) $C_0(x^m)$, the field of fractions of $C_0[x^m]$ (where C_0 denotes the fixed field of C under σ when extended to RC); (*ii*) C_0; (*iii*) $C_0(y)$ where $y = nx^m$ is central and n is in the set of R-normalizing elements of the Martindale ring of quotients.

1. **Preliminaries**.

Let R be a ring and σ an automorphism of R. $R[x, \sigma]$ is the set of all polynomials in x where addition is as usual and multiplication is defined according to the rule $xr = r^\sigma x$, $r \in R$. These operations make $R[x, \sigma]$ into a ring. If R is a prime ring, it is easy to see that $R[x, \sigma]$ is prime. Cohn has determined the center of $R[x, \sigma]$ when R is a division ring.

PROPOSITION 1.1. [1, p. 61] *Let D be a division ring and $\sigma \in Aut(D)$. If no power of σ is inner on D, then the center of $D[x, \sigma]$ is Z_0, the subset of the center of D fixed by σ. If σ^m is inner but no lower power fixes the center of D, then the center of $D[x, \sigma]$ is $Z_0[y]$ where $y = dx^m$ is central.*

To generalize this result to prime rings, we need the notions of X-inner and X-outer automorphisms due to Kharchenko. We begin by summarizing the definition and main properties of the extended centroid and central closure of a prime ring R with 1.

Let $\mu = \{U\}$ be the collection of all nonzero two-sided ideals of R and consider the totality T of all left module homomorphisms $\phi: {}_RU \to {}_RR$, where $U \in \mu$ and U and R are regarded as left R-modules. We write (ϕ, U) for an element of T and define an

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equivalence relation $\sim$ on T as follows: $(\phi, U) \sim (\psi, V)$ if $\phi = \psi$ on some $W \in \mu$ where $W \subseteq U \cap V$. Let $\overline{(\phi, U)}$ denote the equivalence class of (ϕ, U). The Martindale ring of quotients Q is defined to be the set of these equivalence classes. We make Q into a ring as follows:

$$\overline{(\phi, U)} + \overline{(\psi, V)} = \overline{(\phi + \psi, U \cap V)}$$

$$\overline{(\phi, U)}\,\overline{(\psi, V)} = \overline{(\phi \circ \psi, VU)} \text{ (composition acting on the right).}$$

R may be considered a subring of Q via the mapping $a \to \overline{(a_r, R)}$ where a_r is the right multiplication by a acting on R. We state some well-known properties of Q. The proofs can be found in [3].

LEMMA 1.2. *Let Q be as above with center C. Then*
(1) *Q is a prime ring, C is a field and C is the centralizer of R in Q.*
(2) *For any $0 \neq q \in Q$, there exists $U \in \mu$ such that $0 \neq Uq \subseteq R$.*
(3) *Any nonzero left R-submodule of Q intersects R nontrivially.*
(4) *If $\sigma \in$ Aut (R), then σ extends uniquely to an automorphism of Q.*

C is called the extended centroid of R. The pair (ϕ, U) is permissible if ϕ: ${}_R U_R \to {}_R R_R$ is an (R, R)-bimodule homomorphism. C may also be characterized as the set $\{\overline{(\phi, U)} \in Q \mid (\phi, U)$ is permissible$\}$. We may now form the central closure RC of R. RC is a prime ring with center C. A prime algebra over a field F is said to be closed if F is already its extended centroid. We remark that RC is closed over C.

DEFINITION. $\sigma \in$ Aut (R) is X-inner if there exists a unit $q \in Q$ such that $r^\sigma = q^{-1}rq$ for all $r \in R$. In other words, σ is X-inner if its extension to Q is an inner automorphism. Otherwise, σ is X-outer. For any group $G \subseteq$ Aut (R), G is said to be X-outer if the only X-inner automorphism in G is the identity.

LEMMA 1.3. *Suppose G is a subgroup of* Aut (R) *which is X-outer. If $0 \neq a_1, a_2, b_1, b_2 \in R$ and $\sigma_1, \sigma_2 \in G$ such that*

$$a_1 r^{\sigma_1} b_1 + a_2 r^{\sigma_2} b_2 = 0 \textit{ for all } r \in R,$$

then $a_1 = \lambda a_2$ for some $\lambda \in C$ and $\sigma_1 = \sigma_2$.

PROOF. Applying σ_1^{-1} to the above equation and then Lemma 2 of [4], there exists a unit $q \in Q$ with $a_1^{\sigma_1^{-1}} q = a_2^{\sigma_1^{-1}}$ and $q^{-1}rq = r^{\sigma_2 \sigma_1^{-1}}$ for all $r \in R$. Since G is X-outer, $\sigma_1 = \sigma_2$ and hence $q \in C$, proving the lemma. $\square$

In this article we will be concerned with characterizing the extended centroids of related prime rings. The following lemmas will be useful for this purpose.

LEMMA 1.4. *Let R be a prime ring with center Z and extended centroid C. Suppose for any $0 \neq \overline{(\phi, U)} \in C$, there exists $0 \neq u \in U$ such that $u = az_1$ and $u\phi = az_2$ where $z_1, z_2 \in Z$, $a \in R$. Then C is isomorphic to the field of fractions of Z.*

PROOF. Set $\lambda = \overline{(\phi, U)} \in C$. Then $u = az_1$ and $u\phi = u\lambda = az_2$. Thus $u =$

$\lambda^{-1}az_2 = az_1$ implying $(\lambda^{-1}z_2 - z_1)a = 0$. Viewing this equation in RC, $\lambda^{-1}z_2 - z_1 = 0$. Hence $\lambda = z_1^{-1}z_2$ is in the field of fractions of Z. ☐

LEMMA 1.5. *Let $R \subseteq S$ be prime rings with extended centroids C and D respectively. Suppose for any $0 \neq \overline{(\phi, U)} \in C$, there exists $0 \neq u \in U$ such that $u = a\lambda$ and $u\phi = a\beta$ where λ, $\beta \in D$, $a \in R$. Then C is embedded in D. Furthermore, if any nonzero left R-submodule of S intersects R nontrivially, then $C \cong D$.*

PROOF. Let $0 \neq \overline{(\phi, U)} \in C$ and $u \in U$ be as above. Define Φ: $SuS \to S$ by $\sum_i s_iut_i \to \sum_i s_i(u\phi)t_i$. We check Φ is well-defined: Suppose $\sum s_iut_i = 0$. Then $0 = \sum s_ia\lambda t_i = (\sum s_iat_i)\lambda$ (an equation in SD). Hence $\sum s_iat_i = 0$ which implies $0 = \sum s_ia\beta t_i = \sum s_i(u\phi)t_i$. Clearly $\overline{(\Phi, SuS)} \in D$. Note $\overline{(\phi, U)} = \overline{(\phi, RuR)}$ and define T: $C \to D$ by $\overline{(\phi, RuR)} \to \overline{(\Phi, SuS)}$. T is easily seen to be a ring homomorphism and C a field implies T is one-to-one.

Suppose any nonzero left R-submodule of S intersects R nontrivially. Let $0 \neq \overline{(\psi, V)} \in D$ and set $W = V \cap R$. By the assumption, W is a nonzero ideal of R. Since $W\psi$ is a nonzero R-submodule of S, $W\psi \cap R \neq 0$. Hence $U = \{w \in W | w\psi \in R\}$ is a nonzero ideal of R. Setting $\phi = \psi|_U$ we have $\overline{(\phi, U)}T = \overline{(\psi, V)}$. ☐

We are also concerned with the prime ring $(RC)[x, \sigma]$ where σ denotes the extension to RC. An easy application of part (2) of Lemma 1.2 gives

LEMMA 1.6. *Any nonzero left $R[x, \sigma]$-submodule of $(RC)[x, \sigma]$ intersects $R[x, \sigma]$ nontrivially.*

2. $\langle\sigma\rangle$ **X-outer**. In this section we assume $\langle\sigma\rangle$ is X-outer. Let Z denote the center of R and set $Z_o = \{z \in Z | z^\sigma = z\}$.

LEMMA 2.1. *If σ has finite period m, then the center of $R[x, \sigma]$ is $Z_o[x^m]$. If σ has infinite period, the center of $R[x, \sigma]$ is Z_o.*

PROOF. Suppose $f(x) = \sum_i r_ix^i \in$ center $(R[x, \sigma])$. Commuting $f(x)$ with x, we obtain $r_i^\sigma = r_i$ for all i. Commuting $f(x)$ with any $a \in R$, we have $ar_i = r_ia^{\sigma^i}$ for all i. If $r_i \neq 0$, then r_i is a unit in Q and hence σ^i is X-inner. If σ has finite period m, then $m | i$ and $r_i \in Z_o$. If σ has infinite period, then $i = 0$ and $f(x) = r_0 \in Z_o$. ☐

In the next two theorems we determine the extended centroid of $R[x, \sigma]$ in the cases where $\langle\sigma\rangle$ is finite and infinite.

THEOREM 2.2. *Let R be a prime ring with extended centroid C and $\langle\sigma\rangle$ X-outer of finite order m. Let C_o denote the fixed field of C under σ when extended to RC. Then the extended centroid of $R[x, \sigma]$ is isomorphic to $C_o(x^m)$, the field of fractions of $C_o[x^m]$.*

PROOF. We first assume R is closed prime over C and show $R[x, \sigma]$ satisfies the hypotheses of Lemma 1.4. σ extends to an automorphism $\hat{\sigma}$ of $R[x, \sigma]$ as follows: $\sum_i r_ix^i \to \sum_i r_i^\sigma x^i$. To see $\hat{\sigma}$ is multiplicative, we check its effect on monomials:

$$[(rx^j)(sx^k)]^{\hat{\sigma}} = (rs^{\sigma^j}x^{j+k})^{\hat{\sigma}} = r^\sigma s^{\sigma^{j+1}}x^{j+k} = (r^\sigma x^j)(s^\sigma x^k) = (rx^j)^{\hat{\sigma}}(sx^k)^{\hat{\sigma}}.$$

$\hat{\sigma}$ is also of period m. From now on, we write σ for $\hat{\sigma}$.

Let $0 \neq \overline{(\phi, U)} \in$ extended centroid of $R[x, \sigma]$. By considering $0 \neq \bigcap_{i=0}^{m-1} U^{\sigma^i}$, we may assume U is σ-invariant. Let n be the minimal degree of elements of U and set $S = \{f \in U \mid \deg f = n\}$. We claim S contains an element whose coefficients are fixed by σ. Let $V = \{a \in R \mid a$ is the leading coefficient of some $f \in S\}$. Set $V' = V \cup \{0\}$. V' is clearly an ideal of R. Suppose $a \in V$ and $f \in S$ with leading coefficient a. Since U is σ-invariant, $f^\sigma \in S$ with leading coefficient a^σ, showing V' is σ-invariant. By Theorem 3.17 of [3], $\mathrm{tr}_\sigma(V') \neq 0$ where tr denotes the trace relative to $\langle\sigma\rangle$. So there exists $a \in V$ such that $\mathrm{tr}_\sigma(a) \neq 0$. Let

$$f(x) = \sum_{i=k}^{n} a_i x^i$$

with $a_n = a$. Now

$$\mathrm{tr}_\sigma(f(x)) = \sum_{i=k}^{n} (\mathrm{tr}_\sigma(a_i))x^i.$$

Since $f \in U = U^\sigma$, $\mathrm{tr}_\sigma(f(x)) \in U$ and $\mathrm{tr}_\sigma(a) \neq 0$ implies $\mathrm{tr}_\sigma(f(x)) \in S$. Furthermore, the coefficients of $\mathrm{tr}_\sigma(f(x))$ are fixed by σ, proving the claim.

Hence we may choose

$$f(x) = \sum_{i=k}^{n} a_i x^i \in U$$

with $a_n = a$ and $a_i^\sigma = a_i$ for all i. For any $r \in R$,

$$f(x)ra - ar^{\sigma^n}f(x) = \sum_{i=k}^{n-1} (a_i r^{\sigma^i} a - ar^{\sigma^n} a_i)x^i \in U.$$

By minimality, $a_i r^{\sigma^i} a - ar^{\sigma^n} a_i = 0$, $i = k, \ldots, n-1$. By Lemma 1.3, $a_i = \lambda_i a$, $\lambda_i \in C$ and $\sigma^i = \sigma^n$ for all i. Since σ fixes a_i and a, $\lambda_i \in C_0$. Also $\sigma^k = \ldots = \sigma^n$ implies the indices are all congruent to k mod m, i.e., $i = k + q_i m$, $q_i \in Z^+ \cup \{0\}$. Therefore $f(x) = (ax^k)g(x)$ where

$$g(x) = \sum_{i=k}^{n} \lambda_i x^{q_i m} \in C_0[x^m],$$

the center of $R[x, \sigma]$.

Suppose

$$(f(x))\phi = \sum_{j=\ell}^{p} b_j x^j.$$

Using the fact that each $a_i^\sigma = a_i$ and ϕ is bimodule, we obtain $x[(f(x))\phi] = [(f(x))\phi]x$. It follows that $b_j^\sigma = b_j$ for all j. For any $r \in R$, we have

$$0 = (f(x)ra - ar^{\sigma^n}f(x))\phi = [(f(x))\phi]ra - ar^{\sigma^n}[(f(x))\phi]$$
$$= \sum_{j=\ell}^{p} (b_j r^{\sigma^j} a - ar^{\sigma^n} b_j)x^j.$$

Thus we have $b_j r^{\sigma^j} a - ar^{\sigma^n} b_j = 0, j = \ell, \ldots, p$. As before, $b_j = \alpha_j a$ where $\alpha_j \in C_o$, $\sigma^j = \sigma^n$ and $j = \ell + w_j m$, $w_j \in Z^+ \cup \{0\}$. Hence $(f(x))\phi = (ax^\ell)h(x)$ where

$$h(x) = \sum_{j=\ell}^{p} \alpha_j x^{w_j m} \in C_o[x^m].$$

Since $\sigma^k = \sigma^n = \sigma^\ell$, we have $k \equiv \ell \pmod{m}$. If $k > \ell$, then $k = \ell + tm$, $t \in Z^+$. In this case, $f(x) = (ax^\ell)q(x)$ for some $q(x) \in C_o[x^m]$. Similarly, if $\ell > k$, $(f(x))\phi = (ax^k)p(x)$ for some $p(x) \in C_o[x^m]$. By Lemma 1.4, the extended centroid of $R[x,\sigma]$ is isomorphic to $C_o(x^m)$.

Now suppose R is a prime ring and let σ denote the extension to RC. One can show $\langle\sigma\rangle$ is X-outer on RC of order m. Since RC is closed prime over C, the extended centroid of $(RC)[x,\sigma]$ is isomorphic to $C_o(x^m)$. Let $0 \neq \overline{(\phi, U)} \in$ extended centroid of $R[x,\sigma]$. By the above argument, there exists $0 \neq f(x) \in U$ such that $f(x) = a(x)z_1(x)$ and $(f(x))\phi = a(x)z_2(x)$ where $z_1(x)$, $z_2(x) \in C_o[x^m]$ (the center of $(RC)[x,\sigma]$), $a(x) \in R[x,\sigma]$. By Lemmas 1.5 and 1.6, we may conclude $R[x,\sigma]$ and $(RC)[x,\sigma]$ have isomorphic extended centroids, completing the proof of the theorem. ☐

THEOREM 2.3. *Let R be a prime ring with extended centroid C. If $\langle\sigma\rangle$ is X-outer and infinite, then the extended centroid of $R[x,\sigma]$ is isomorphic to C_o.*

PROOF. Assume R is closed prime. Let $0 \neq \overline{(\phi, U)} \in$ extended centroid of $R[x,\sigma]$. Choose

$$0 \neq f(x) = \sum_{i=k}^{n} a_i x^i \in U, \qquad a_n = a,$$

of minimal degree. We claim $a_i = 0$ for $i = k, \ldots, n-1$. Suppose $a_j \neq 0$ for some j such that $k \leq j \leq n-1$. There exists $b \in R$ such that $b^{\sigma^n} = a$. For any $r \in R$,

$$f(x)rb - ar^{\sigma^n}f(x) = \sum_{i=k}^{n-1} (a_i r^{\sigma^i} b^{\sigma^i} - ar^{\sigma^n} a_i)x^i \in U.$$

By minimality, $a_j r^{\sigma^j} b^{\sigma^j} - ar^{\sigma^n} a_j = 0$. By Lemma 1.3, $\sigma^j = \sigma^n$ and σ of infinite period implies $j = n$, a contradiction. Hence $f(x) = ax^n$. A similar argument gives $(f(x))\phi = \lambda ax^n$ for some $\lambda \in C$.

We now show $\lambda \in C_o$. Since R is prime, we may choose $r \in R$ such that $ar^{\sigma^n}a^{\sigma^n} \neq 0$. ϕ bimodule implies $f(x)r[(f(x))\phi] = [(f(x))\phi]rf(x)$. So we have $(ax^n)r(\lambda ax^n) = (\lambda ax^n)r(ax^n)$ which gives

$$\lambda^{\sigma^n} ar^{\sigma^n} a^{\sigma^n} = \lambda ar^{\sigma^n} a^{\sigma^n}$$

or

$$(\lambda^{\sigma^n} - \lambda)ar^{\sigma^n}a^{\sigma^n} = 0.$$

Hence $\lambda^{\sigma^n} = \lambda$. Now choose $r \in R$ such that $ar^{\sigma^{n+1}}a^{\sigma^{n+1}} \neq 0$. As above, $f(x)xr[(f(x))\phi] = [(f(x))\phi]xrf(x)$ yields $\lambda^{\sigma^{n+1}} = \lambda$ and so $\lambda \in C_0$.

Therefore $f(x) = ax^n$ and $(f(x))\phi = (ax^n)\lambda$ where $\lambda \in C_0$ (the center of $R[x,\sigma]$). By Lemma 1.4, the extended centroid of $R[x,\sigma]$ is isomorphic to C_0.

Now suppose R is a prime ring and let σ denote the extension to RC. One can show $\langle\sigma\rangle$ is X-outer and infinite. Since RC is closed prime, the extended centroid of $(RC)[x,\sigma]$ is isomorphic to C_0. As in Theorem 2.2, $R[x,\sigma]$ and $(RC)[x,\sigma]$ have isomorphic extended centroids. ☐

3. **σ^m X-inner**. To treat the case where σ^m is X-inner for some m, we need the notion of the normal closure of a prime ring R. Set $N = \{n \in Q \mid nR = Rn\}$. We remark that if $0 \neq n \in N$, then n is invertible and $n^{-1} \in N$. The normal closure RN of R in Q is the subring of Q generated by R and N, i.e.,

$$RN = \left\{\sum_i r_i n_i \in Q \mid r_i \in R,\ n_i \in N\right\}.$$

LEMMA 3.1. *RN has the following properties* [3]:
(1) *RN is a closed prime ring with extended centroid C.*
(2) *For any $0 \neq q \in RN$, there exists a nonzero ideal I of R such that $0 \neq Iq \subseteq R$ and $0 \neq qI \subseteq R$.*
(3) *If $\sigma \in$ Aut (R), then its extension to Q restricts to an automorphism of RN.*

LEMMA 3.2. *For any $\sigma \in$ Aut (R), $R[x,\sigma]$ and $(RN)[x,\sigma]$ have isomorphic extended centroids.*

PROOF. Let E and F be the extended centroids of $R[x,\sigma]$ and $S = (RN)[x,\sigma]$ respectively. Choose $0 \neq \overline{(\phi, U)} \in E$. Define Φ: $SUS \to S$ by

$$\sum_i f_i(x)u_i(x)g_i(x) \to \sum_i f_i(x)[(u_i(x))\phi]g_i(x).$$

Suppose

$$\sum_i f_i(x)u_i(x)g_i(x) = 0$$

where

$$f_i(x) = \sum_j a_{ij}x^j$$

and

$$g_i(x) = \sum_k b_{ik}x^k.$$

By Lemma 3.1, there exists a nonzero ideal I of R such that $0 \neq Ia_{ij} \subseteq R$ and $0 \neq b_{ik}I \subseteq R$. For each i, set $J_i = \bigcap_k I^{(\sigma^k)^{-1}}$ where k runs through the sum in $g_i(x)$. Then $0 \neq g_i(x) J_i \subseteq R[x, \sigma]$. Set $J = \bigcap_i J_i$. J is a nonzero ideal of R and $0 \neq g_i(x) J \subseteq R[x, \sigma]$ for all i. Hence we have

$$I\Big(\sum_i f_i(x)[(u_i(x))\phi]g_i(x)\Big)J = 0$$

which implies

$$\sum_i f_i(x)[(u_i(x))\phi]g_i(x) = 0,$$

proving Φ is well-defined. Define T: $E \to F$ by$(\overline{\phi, U}) \to (\overline{\Phi, SUS})$. The usual argument shows T is a ring isomorphism. $\square$

For the remainder of this section, let m be the least positive integer such that σ^m is X-inner. We assume no lower power of σ is the identity on C. Note that by a result of Kharchenko [3, p. 48], if R satisfies a GPI, this assumption is superfluous.

LEMMA 3.3. *The center of* $(RN)[x, \sigma]$ *is* $C_o[y]$ *where* $y = nx^m$ *is central.*

PROOF. Since σ^m is X-inner, there exists $n \in N$ such that $a^{\sigma^m} = n^{-1}an$ for all $a \in Q$. It is easy to verify that $n^\sigma n^{-1} = \lambda \in C$ and the norm of λ (i.e., $\lambda^{\sigma^{m-1}}\lambda^{\sigma^{m-2}} \ldots \lambda^\sigma\lambda$) equals 1. Since σ induces an automorphism of period m on C, Hilbert's Theorem 90 implies $\lambda = \mu^\sigma\mu^{-1}$, $\mu \in C$. Setting $n_1 = n\mu^{-1}$, observe that $n_1^\sigma = n_1$ and $a^{\sigma^m} = n_1^{-1}an_1$ for all $a \in Q$. Hence without loss of generality, we may assume $n^\sigma = n$. Set $y = nx^m$. Then $xy = xnx^m = n^\sigma x^{m+1} = nx^m x = yx$ and $ya = nx^m a = n(n^{-1}an)x^m = ay$ for any $a \in RN$. So $C_o[y] \subseteq$ center $((RN)[x, \sigma])$.

Suppose

$$f(x) = \sum_i a_i x^i \in \text{center } ((RN)[x, \sigma]).$$

As usual, $a_i^\sigma = a_i$ and $sa_i = a_i s^{\sigma^i}$ for all i and $s \in RN$. If $a_i \neq 0$, then σ^i is X-inner and hence $i = mq_i$ where $q_i \in Z^+ \cup \{0\}$. Now $a_i^{-1}sa_i = s^{\sigma^{mq_i}} = n^{-q_i}sn^{q_i}$ for all $s \in RN$ implying $a_i = \lambda_i n^{q_i}$, $\lambda_i \in C$. Since a_i and n are fixed by σ, $\lambda_i \in C_o$. Rewriting we have

$$f(x) = \sum_i \lambda_i (nx^m)^{q_i} = \sum_i \lambda_i y^{q_i} \in C_o[y]. \quad \square$$

The proof of Lemma 3.3 follows Cohn [1]. We now prove the main result of this section.

THEOREM 3.4. *Let R be a prime ring with extended centroid C and m the least positive integer such that* σ^m *is X-inner. If no smaller power of* σ *fixes C, then the extended centroid of* $R[x, \sigma]$ *is isomorphic to* $C_o(y)$ *where* $y = nx^m$ *is central.*

PROOF. For any $a \in Q$, $a^{\sigma^m} = n^{-1}an$ where $n \in N$ and $n^\sigma = n$. By Lemma 3.2, it suffices to show that the extended centroid of $(RN)[x, \sigma]$ is isomorphic to $C_o(y)$. Let

$0 \neq (\overline{\phi, U}) \in$ extended centroid of $(RN)[x, \sigma]$ and choose

$$f(x) = \sum_{i=k}^{p} a_i x^i \in U$$

of minimal degree. There exists $b \in RN$ such that $b^{\sigma^p} = a_p = a$. For any $s \in RN$, $f(x)sb - as^{\sigma^p} f(x) \in U$ of smaller degree than f implying $as^{\sigma^p} a_i - a_i s^{\sigma^i} b^{\sigma^i} = 0$, $i = k, \ldots, p - 1$. Applying $(\sigma^p)^{-1}$ to this equation and Lemma 2 of [4], there exists $n_i \in N$ such that $a_i = an_i^{\sigma^p}$ and $n_i^{-1} s n_i = s^{\sigma^i}(\sigma^p)^{-1}$. Thus $\sigma^i(\sigma^p)^{-1}$ is X-inner and $\sigma^i = \sigma^p$ on C. It follows that $\sigma^i = \sigma^k$ on C and since $\sigma \in \operatorname{Aut}(C)$ is of period m, $i = k + q_i m$ where $q_i \in Z^+ \cup \{0\}$, $i = k, \ldots, p$. Now $\sigma^i(\sigma^p)^{-1} = \sigma^{(q_i - q_p)m}$ implies $n_i^{-1} s n_i = n^{q_p - q_i} s n^{q_i - q_p}$ for all $s \in RN$. Therefore $n_i = \alpha_i n^{q_i - q_p}$, $\alpha_i \in C$, and $a_i = a(\alpha_i n^{q_i - q_p})^{\sigma^p} = a\beta_i n^{q_i - q_p}$, $\beta_i \in C$. Setting $y = nx^m$ we have

$$f(x) = \sum_{i=k}^{p} a\beta_i n^{q_i - q_p} x^{k + q_i m} = an^{-q_p} x^k \sum_{i=k}^{p} \beta_i^{(\sigma^k)^{-1}} n^{q_i} x^{q_i m} = an^{-q_p} x^k \sum_{i=k}^{p} \gamma_i y^{q_i}$$

for some $\gamma_i \in C$.

Suppose

$$(f(x))\phi = \sum_{j=\ell}^{r} b_j x^j.$$

For any $s \in RN$,

$$0 = (f(x)sb - as^{\sigma^p} f(x))\phi = [(f(x))\phi]sb - as^{\sigma^p}[(f(x))\phi]$$
$$= \sum_{j=\ell}^{r} (b_j s^{\sigma^j} b^{\sigma^j} - as^{\sigma^p} b_j) x^j$$

which implies $as^{\sigma^p} b_j - b_j s^{\sigma^j} b^{\sigma^j} = 0$, $j = \ell, \ldots, r$. As before, there exists $m_j \in N$ such that $b_j = am_j^{\sigma^p}$, $m_j^{-1} s m_j = s^{\sigma^j(\sigma^p)^{-1}}$, and $\sigma^j = \sigma^p$ on C. Thus $\sigma^k = \ldots = \sigma^p = \sigma^\ell = \ldots = \sigma^r$ on C. So $j = \ell + w_j m$ for all j and $p = \ell + tm$ where $w_j \in Z^+ \cup \{0\}$, $t \in Z$. Hence $m_j^{-1} s m_j = n^{t - w_j} s n^{w_j - t}$ and so $m_j = \delta_j n^{w_j - t}$, $\delta_j \in C$. As above, $b_j = a\lambda_j n^{w_j - t}$ and

$$(f(x))\phi = an^{-t} x^\ell \sum_{j=\ell}^{r} \rho_j y^{w_j}$$

for some $\rho_j \in C$.

Assume $k > \ell$. Now $p = \ell + tm = k + q_p m$ implies $k = \ell + (t - q_p)m$, $t - q_p \in Z^+$. Then

$$f(x) = an^{-q_p} x^{\ell + (t - q_p)m} \sum_{i=k}^{p} \gamma_i y^{q_i} = an^{-q_p} x^\ell \sum_{i=k}^{p} \nu_i x^{(t - q_p)m} y^{q_i}$$
$$= an^{-q_p} x^\ell \sum_{i=k}^{p} \nu_i n^{q_p - t} y^{t - q_p + q_i} = an^{-t} x^\ell \sum_{i=k}^{p} \nu_i y^{t - q_p + q_i}$$

for some $\nu_i \in C$.

An analogous argument works for $\ell > k$. Summarizing, we have shown that $f(x) =$

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$an^ix^jg_1(y)$ and $(f(x))\phi = an^ix^jg_2(y)$ for some i, j and $g_1(y), g_2(y) \in C[y]$. We are not yet in a position to apply Lemma 1.4 since $g_1(y)$ and $g_2(y)$ are not necessarily in $C_o[y]$, the center of $(RN)[x, \sigma]$.

Now σ may be extended to $C[y]$ by mapping y to y. We claim $g_1(y)^\sigma g_2(y) = g_1(y)g_2(y)^\sigma$. First note that $C[y]$ centralizes RN in S = central closure of $(RN)[x, \sigma]$. Also if R is any prime ring, $\sigma \in \text{Aut}(R)$, and $p(x)Rq(x) = 0$ for $p(x), q(x) \in R[x, \sigma]$, then $p(x) = 0$ or $q(x) = 0$. For any $s \in RN, f(x)s[(f(x))\phi] = [f(x))\phi]sf(x)$ implies

$$(an^ix^j)s(an^ix^j(g_1(y)^{\sigma^j}g_2(y) - g_1(y)g_2(y)^{\sigma^j})) = 0.$$

Since $an^ix^j \neq 0$ the above remarks give $g_1(y)^{\sigma^j}g_2(y) - g_1(y)g_2(y)^{\sigma^j} = 0$. By using xs, we similarly obtain $g_1(y)^{\sigma^{j+1}}g_2(y) - g_1(y)g_2(y)^{\sigma^{j+1}} = 0$. The claim follows from these two equations.

Working in S, we have $f(x) = an^ix^jg_1(y)$ and $f(x)\lambda = an^ix^jg_2(y)$ for $\lambda = (\overline{\phi, U}) \in$ extended centroid of $(RN)[x, \sigma]$. Hence $an^ix^j(g_1(y) - \lambda^{-1}g_2(y)) = 0$. If $g_1(y) - \lambda^{-1}g_2(y) \neq 0$, there exists a nonzero ideal V of $(RN)[x, \sigma]$ such that $0 \neq (g_1(y) - \lambda^{-1}g_2(y))V \subseteq (RN)[x, \sigma]$. Thus

$$(an^ix^j)(RN)(g_1(y) - \lambda^{-1}g_2(y))V = 0$$

implies

$$(g_1(y) - \lambda^{-1}g_2(y))V = 0,$$

a contradiction. Therefore $g_2(y) = \lambda g_1(y)$.

Note that the extension of σ to $C[y]$ agrees with the extension of σ to S when restricted to $C[y]$. Also the fixed ring of $\langle\sigma\rangle$ acting on $C[y]$ is $C_o[y]$.

Finally $g_1(y)^\sigma g_2(y) = g_1(y)g_2(y)^\sigma$ and $g_2(y) = \lambda g_1(y)$ imply $(\lambda - \lambda^\sigma)g_1(y)g_1(y)^\sigma = 0$ and so $\lambda^\sigma = \lambda$. Since any nonzero ideal of $C[y]$ intersects $C_o[y]$ nontrivially, there exists $0 \neq h(y) \in C[y]$ such that $0 \neq g_2(y)h(y) = \lambda g_1(y)h(y) \in C_o[y]$. It follows that $g_1(y)h(y) \in C_o[y]$. Now $0 \neq f(x)h(y) = an^ix^jg_1(y)h(y) \in U$ and $(f(x)h(y))\phi = [(f(x))\phi]h(y) = an^ix^jg_2(y)h(y)$ where $g_1(y)h(y), g_2(y)h(y) \in C_o[y]$. By Lemmas 1.4 and 3.3, the extended centroid of $(RN)[x, \sigma]$ is isomorphic to $C_o(y)$. ☐

We end this paper with a brief discussion of the remaining case: σ^m is X-inner and some lower power of σ is the identity on C. The primary difficulty seems to be in determining the center of $(RN)[x, \sigma]$. We can no longer use Hilbert's Theorem 90 to find an n fixed by σ that determines σ^m as an inner automorphism of Q. As already noted, if R satisfies a GPI, this remaining case cannot occur.

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ON THE PONTRJAGIN ALGEBRA OF A CERTAIN CLASS OF FLAGS OF FOLIATIONS

BY

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ABSTRACT. Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ be mutually orthogonal distributions on $\mathcal{M}$ of dimensions p_1, p_2, p_3 such that $p_1 + p_2 + p_3 = \dim \mathcal{M}$. We assume that $\mathcal{V}_1$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$ are integrable and that all the geodesics of $\mathcal{M}$ with initial tangent vector in $\mathcal{V}_2$ remain tangent to $\mathcal{V}_2$. Then, we prove that $\mathrm{Pont}^k(\mathcal{V}_2 \oplus \mathcal{V}_3) = 0$ for $k > p_2 + 2p_3$, where $\mathrm{Pont}^k(\mathcal{V}_2 \oplus \mathcal{V}_3)$ is the subspace of the Pontrjagin algebra of $\mathcal{V}_2 \oplus \mathcal{V}_3$ generated by forms of degree k.

1. **Introduction**. Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\mathcal{V}$ a distribution on $\mathcal{M}$. If $\mathcal{V}$ is integrable, then Bott's theorem [1] says that $\mathrm{Pont}^k(\mathcal{V}^\perp) = 0$ for $k > 2 \dim (\mathcal{V}^\perp)$, where $\mathrm{Pont}^k(\mathcal{V}^\perp)$ is the subspace of the Pontrjagin algebra of $\mathcal{V}^\perp$ generated by forms of degree k. On the other hand, Pasternack [7] has proved that if $\mathcal{V}$ is integrable and the metric is bundle-like with regard to the corresponding foliation, then $\mathrm{Pont}^k(\mathcal{V}^\perp) = 0$ for $k > \dim (\mathcal{V}^\perp)$. In this paper we prove a result on flags of two foliations that turns out to be a generalization of the foregoing ones. In fact, let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ be mutually orthogonal distributions of dimensions p_1, p_2, p_3 on $(\mathcal{M}, g)$, such that the tangent bundle $T\mathcal{M} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$, where we identify each distribution with the vector bundle determined by it. We assume that $\mathcal{V}_1$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$ are integrable and that $\mathcal{V}_2$ is a totally geodesic distribution, in the sense that any geodesic of $\mathcal{M}$ with initial tangent vector in $\mathcal{V}_2$ remains tangent to $\mathcal{V}_2$. Then, $\mathrm{Pont}^i(\mathcal{V}_2)\,\mathrm{Pont}^j(\mathcal{V}_3) = 0$ for $j > 2p_3$ or $i + j > p_2 + 2p_3$. This condition is stronger than the one obtained by L. A. Cordero and X. Masa in [2] for flags of two foliations (subfoliations) without the additional hypothesis on $\mathcal{V}_2$. In [2] an extensive study of the characteristic classes of subfoliations is also carried out.

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2. **The generalized second connection**. In order to find a suitable expression for the Pontrjagin polynomials of the bundle $\mathcal{V}_2 \oplus \mathcal{V}_3$, we define what we call the generalized second connection $\tilde{\nabla}$. Since its construction is valid for an arbitrary number of distributions, we shall consider, within this section, k mutually orthogonal distributions $\mathcal{V}_1, \ldots, \mathcal{V}_k$ on a Riemannian manifold $(\mathcal{M}, g)$ such that $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k = T\mathcal{M}$.

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If v_i is the projector of $T\mathcal{M}$ onto $\mathcal{V}_i$, $1 \leq i \leq k$, and ∇ is the Levi-Civita connection, we set

$$\bar{\nabla}_{A_i} B_i = v_i(\nabla_{A_i} B_i) \text{ for } 1 \leq i \leq k$$

and

$$\bar{\nabla}_{A_i} B_j = v_j[A_i, B_j] \text{ for } 1 \leq i, j \leq k \text{ and } i \neq j.$$

Throughout the paper the subscripts of the vector fields, if any, indicate the distribution to which they belong.

It is clear that $\bar{\nabla}$ is a well-defined connection on $T\mathcal{M}$ which satisfies

(*i*) For all i, $1 \leq i \leq k$, $\bar{\nabla} v_i = 0$, or, equivalently, the restriction of $\bar{\nabla}$ to each $\mathcal{V}_i$ is a connection in $\mathcal{V}_i$.

(*ii*) $A_i(g(B_i, C_i)) = g(\bar{\nabla}_{A_i} B_i, C_i) + g(B_i, \bar{\nabla}_{A_i} C_i)$, since the $\mathcal{V}_i$'s are mutually orthogonal and $\nabla g = 0$.

(*iii*) If T is the torsion tensor of $\bar{\nabla}$, i.e., $T(M,N) = \bar{\nabla}_M N - \bar{\nabla}_N M - [M,N]$ for all M, $N \in \mathfrak{X}(\mathcal{M})$, then $g(T(A_i, M), B_i) = 0$ for all $M \in \mathfrak{X}(\mathcal{M})$ and A_i, $B_i \in \mathcal{V}_i$, $1 \leq i \leq k$, as a consequence of the symmetry of ∇.

Furthermore, we have

THEOREM 1. *Let* $(\mathcal{M}, g)$ *be a Riemannian manifold and* $\mathcal{V}_1, \ldots, \mathcal{V}_k$ *mutually orthogonal distributions on* $\mathcal{M}$ *such that* $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k = T\mathcal{M}$. *There exists a unique connection in* $T\mathcal{M}$ *satisfying* (*i*), (*ii*) *and* (*iii*).

PROOF. Only uniqueness has to be proved. Assume ∇^* is such a connection in $T\mathcal{M}$ and let T^* be its torsion tensor; then, for any A_i, B_i, C_j with $i \neq j$, we have

$$\begin{aligned} 0 = g(T^*(A_i, C_j), B_i) &= g(\nabla^*_{A_i} C_j - \nabla^*_{C_j} A_i - [A_i, C_j], B_i) \\ &= -g(\nabla^*_{C_j} A_i + v_i[A_i, C_j], B_i), \end{aligned}$$

and since B_i is an arbitrary vector field in $\mathcal{V}_i$, we get

$$\nabla^*_{C_j} A_i = -v_i[A_i, C_j] = \bar{\nabla}_{C_j} A_i.$$

On the other hand,

$$g(T^*(A_i, B_i), C_i) = 0$$

implies

$$\nabla^*_{A_i} B_i - \nabla^*_{B_i} A_i = v_i[A_i, B_i].$$

Now, the Riemannian connection satisfies

$$\begin{aligned} 2g(\nabla_{A_i} B_i, C_i) = A_i(g(B_i, C_i)) &+ B_i(g(A_i, C_i)) - C_i(g(A_i, B_i)) \\ &+ g([C_i, A_i], B_i) + g([C_i, B_i], A_i) + g([A_i, B_i], C_i). \end{aligned}$$

and a substitution in this formula gives $g(\nabla_{A_i} B_i, C_i) = g(\nabla^*_{A_i} B_i, C_i)$, whence

$$\nabla^*_{A_i} B_i = v_i(\nabla_{A_i} B_i) = \bar{\nabla}_{A_i} B_i. \; //$$

It is easily seen that the semicanonical connection defined by Vaisman [9] satisfies the conditions of theorem 1, so that both connections are the same.

Let S be the curvature tensor of $\tilde{\nabla}$, i.e.

$$S(L, M, N, O) = g(\tilde{\nabla}_M \tilde{\nabla}_L N - \tilde{\nabla}_L \tilde{\nabla}_M N + \tilde{\nabla}_{[L, M]} N, O)$$

for all $L, M, N, O \in \mathfrak{X}(\mathcal{M})$.

As a consequence of properties (i), (ii) and (iii) of $\tilde{\nabla}$, we have

Proposition 2. *Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ mutually orthogonal distributions on $\mathcal{M}$ such that $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k = T\mathcal{M}$. If S is the curvature tensor of $\tilde{\nabla}$, then, for $i \neq j$:*

$$(a) \qquad S(M, N, A_i, B_j) = S(M, N, B_j, A_i) = 0,$$

$$(b) \qquad S(A_i, B_i, C_j, D_j) = -g([A_i, v_j[B_i, C_j]], D_j) + g([B_i, v_j[A_i, C_j]], D_j) + g(\nabla_{C_j} v_j[A_i, B_i], D_j) + g([[A_i, B_i], C_j], D_j),$$

$$(c) \qquad S(A_i, B_j, C_j, D_j) - S(A_i, C_j, B_j, D_j) = -\sum_{h \neq j} g([v_h[B_j, C_j], A_i], D_j),$$

$$(d) \qquad S(A_i, B_i, C_i, D_i) + S(A_i, B_i, D_i, C_i) = \sum_{h \neq i} g(v_h[A_i, B_i], \nabla_{C_i} D_i + \nabla_{D_i} C_i),$$

$$(e) \qquad \underset{A_i, B_i, C_i}{\mathfrak{S}} S(A_i, B_i, C_i, D_i) = 0. \quad \|$$

Corollary 3. *If one of the three following conditions is satisfied for $i \neq j$*

(a) $\mathcal{V}_j$ is integrable,

(b) $\oplus_{h \neq j} \mathcal{V}_h$ is integrable,

(c) For every $h \neq j$, $\mathcal{V}_h \oplus \mathcal{V}_i$ is integrable;

then,

$$S(A_i, B_j, C_j, D_j) = S(A_i, C_j, B_j, D_j). \quad \|$$

Corollary 4. *If $\mathcal{V}_i$ is integrable, then*

$$(a) \qquad S(A_i, B_i, C_i, D_i) = -S(A_i, B_i, D_i, C_i)$$

$$(b) \qquad S(A_i, B_i, C_i, D_i) = R'(A_i, B_i, C_i, D_i)$$

where R' is the curvature tensor of the Levi-Civita connection of the leaves of $\mathcal{V}_i$.

If, further, there exists j, $i \neq j$, such that

(1) $\oplus_{h \neq j} \mathcal{V}_h$ is integrable, or

(2) for every h, $h \neq j$, $\mathcal{V}_i \oplus \mathcal{V}_h$ is integrable,

then,

$$(c) \qquad S(A_i, B_i, C_j, D_j) = 0. \quad \|$$

Henceforth, we will consider an additional condition on one of the distributions, namely, DEFINITION [3], [6]. *A distribution* $\mathcal{V}$ *will be said to be of type* D_1 *if* $(\nabla_A v)A = 0$ *for all* A *in* $\mathcal{V}$*; where* v *is the orthogonal projection onto* $\mathcal{V}$.

THEOREM 5. *A p-dimensional distribution* $\mathcal{V}$ *in an n-dimensional Riemannian manifold* $(\mathcal{M}, g)$ *is of type* D_1 *if and only if every geodesic of* $\mathcal{M}$ *with initial tangent vector in* $\mathcal{V}$ *remains tangent to* $\mathcal{V}$.

PROOF. Let σ be a curve in $\mathcal{M}$, and let $\alpha = \dot{\sigma}$ be its tangent vector field, which is a curve in $T\mathcal{M}$. Then, σ is a geodesic if and only if α is an integral curve of the spray ξ associated to ∇. Let $\{E_1, \ldots, E_n\}$ be a local frame defined in an open set U of $\mathcal{M}$ such that $E_1, \ldots, E_p \in \mathcal{V}$ and $E_{p+1}, \ldots, E_n \in \mathcal{V}$. This frame defines a trivialization $TU \cong U \times \mathbb{R}^n$. Under this identification the field ξ can be written as

$$\xi = \sum_i p^i E_i - \sum_{i,j,k} p^i p^j \theta^k(\nabla_{E_i} E_j) \frac{\partial}{\partial p^k},$$

where the p^i's $(1 \leq i \leq n)$ are the standard coordinates on $\mathbb{R}^n$, and $\{\theta^k\}$ is the dual frame of $E_1, \ldots, E_n$. On the other hand, the condition of $\mathcal{V}$ being of type D_1 is clearly equivalent to the fact that, for all $a, b = 1, \ldots, p$ and $u = p + 1, \ldots, n$,

$$(*) \qquad \theta^u(\nabla_{E_a} E_b) + \theta^u(\nabla_{E_b} E_a) = 0.$$

Further, the subbundle of $T\mathcal{M}$, determined by $\mathcal{V}$, which will be eqally denoted by $\mathcal{V}$, is a regular submanifold of $T\mathcal{M}$, and a tangent vector to $T\mathcal{M}$ is tangent to $\mathcal{V}$ if and only if it is a linear combination of $E_1, \ldots, E_p, \partial/\partial p^1, \ldots, \partial/\partial p^p$ (under the above identification). So, (*) is equivalent to the fact that the restriction of ξ to $\mathcal{V}$ be tangent to $\mathcal{V}$; in other words, the fact that the integral curves of ξ with initial values in $\mathcal{V}$ be contained in $\mathcal{V}$, whence the result follows. //

Notice that if $\mathcal{F}$ is a foliation in $(\mathcal{M}, g)$, then $\mathcal{F}$ is of type D_1 if and only if the metric is bundle-like [6], and thus, theorem 5 generalizes Reinhart's result about geodesics in a foliated manifold with bundle-like metric [8].

EXAMPLE. Consider the sphere S^{4n+3} and the Hopf fibration onto the quaternionic n-space $S^{4n+3} \to \mathbb{H}^n$. Let $\mathcal{V}$ be the distribution determined by the vector spaces tangent to the fibres, and let $\mathcal{H} = \mathcal{V}^\perp$. On the other hand, consider $S^r \times \mathbb{R}^t$, where $r + t = 4m + 3$ for some integer m. As an orientable hypersurface in $\mathbb{R}^{4m+4}$, $S^r \times \mathbb{R}^t$ has a global normal vector field N. If J_1, J_2, J_3 are the three canonical almost-complex structures on $\mathbb{R}^{4m+4} = \mathbb{H}^m$, then J_1N, J_2N, J_3N are vector fields tangent to $S^r \times \mathbb{R}^t$ which span a 3-dimensional distribution $\mathcal{V}'$. Then, on $S^{4n+3} \times S^r \times \mathbb{R}^t$, $\mathcal{H} \oplus \mathcal{V}'$ is a non-integrable distribution of type D_1, whose orthogonal complement is not integrable. For more details and further examples, see [5].

PROPOSITION 6. *If* $\mathcal{V}_i$ *is of type* D_1*, then*

(a) $S(M, N, A_i, B_i) = -S(M, N, B_i, A_i)$

(b) The curvature tensors S and R of $\bar{\nabla}$ *and* ∇*, respectively, agree when acting upon arguments of* $\mathcal{V}_i$ *if and only if* $\mathcal{V}_i$ *is integrable with totally geodesic leaves.*

The proof follows by computation from properties (*i*), (*ii*), (*iii*) and the following.

LEMMA 7. *If* $\mathcal{V}_i$ *is of type* D_1 and $M \in \oplus_{h \neq i} \mathcal{V}_h$, *then*

$$M(g(A_i, B_i)) = g([M, A_i], B_i) + g(A_i, [M, B_i]),$$

or, equivalently,

$$(\mathcal{L}_M g)(A_i, B_i) = 0 \ \|$$

where $\mathcal{L}$ *denotes the Lie derivative.* $\|$

COROLLARY 8. *If* $\mathcal{V}_i$ *is of type* D_1 *and if further one of conditions* (a), (b), (c) *of corollary 3 is satisfied, then* $S(A_j, B_i, C_i, D_i) = 0$. $\|$

3. **The obstruction theorem**. Let E be a vector bundle on $\mathcal{M}$, which we can assume as a subbundle of $T\mathcal{M}$, with fibre $\mathbb{R}^r$ and D a connection in E. If we consider the associated principal bundle we have the Weyl homomorphism [4]

$$w\colon I(\mathrm{Gl}(r, \mathbb{R})) \to H^*(\mathcal{M}, \mathbb{R})$$

where $I(\mathrm{Gl}(r, \mathbb{R}))$ $(= \sum I^k(\mathrm{Gl}(r, \mathbb{R})))$ is the graded algebra of ad-$\mathrm{Gl}(r, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{R})$, the Lie algebra of $\mathrm{Gl}(r, \mathbb{R})$. The image of this homomorphism is the algebra of characteristic classes of E and will be denoted by Pont (E). For each k, $w(I^k(\mathrm{Gl}(r, \mathbb{R})))$ is the space of characteristic classes of degree $2k$, and will be denoted by $\mathrm{Pont}^{2k}(E)$.

The algebra Pont (E) is spanned by the classes $w(f_0)$, $w(f_1)$, $\ldots$, $w(f_{2m})$ with $r = 2m$ or $2m + 1$, where $f_0, f_1, \ldots, f_r$ are ad-$\mathrm{Gl}(r, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{R})$, defined by

$$\det\left(\lambda I_r - \frac{1}{2\pi} X\right) = \sum_{k=0}^{r} f_k(X) \lambda^{r-k}$$

for all $X \in \mathfrak{gl}(r, \mathbb{R})$, where I_r is the identity matrix of dimension $r \times r$. For each $k \in \{0, \ldots, m\}$ the class $w(f_{2k}) \in H^{4k}(\mathcal{M}, \mathbb{R})$ is the k-th Pontrjagin class $p_k(E)$ of the bundle E.

From the relations between the curvature tensor S and the curvature form Ω of D, it follows that if f is an invariant homogeneous polynomial of degree k, then there exists a representative α in $w(f)$ such that, if $\{Y_j\}_{j=1,\ldots,r}$ is a local orthonormal frame of E and $X_1, \ldots, X_{2k}$ are linearly independent vector fields on $\mathcal{M}$, then $\alpha(X_1, \ldots, X_{2k})$ is a homogeneous polynomial of degree k in the variables $\{S(X_1, X_m, Y_i, Y_j)\}$, $i, j \in \{1, \ldots, r\}$, $m, 1 \in \{1, \ldots, 2k\}$, so that, within each monomial, the first two arguments of each factor perform a permutation of $X_1, \ldots, X_{2k}$.

Now, let $(\mathcal{M}, g)$ be a Riemannian manifold and let $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ be three orthogonal distributions on $\mathcal{M}$ of dimensions p_1, p_2, p_3, respectively, such that $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 = T\mathcal{M}$, $\mathcal{V}_1$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$ are integrable, and, v_i being the projector of $T\mathcal{M}$ onto $\mathcal{V}_i$, $(i = 1, 2, 3)$

$$(\nabla_{A_2} v_2) A_2 = 0$$

for all $A_2 \in \mathcal{V}_2$, where ∇ is the Levi-Civita connection.

Then, we have

LEMMA 9.

$$\begin{aligned} S(A_1, B_1, C_2, D_2) &= S(A_1, B_1, C_3, D_3) \\ &= S(A_1, B_2, C_2, D_2) \\ &= S(A_1, B_2, C_3, D_3) = 0. \end{aligned}$$

PROOF. It follows from corollary 4 (c), that $S(A_1, B_1, C_3, D_3) = 0$. From proposition $2(b)$, and the integrability of $\mathcal{V}_1 \oplus \mathcal{V}_2$, we get $S(A_1, B_1, C_2, D_2) = 0$. $S(A_1, B_2, C_2, D_2) = 0$ follows from proposition $2(c)$, and $6(a)$. Finally,

$$\begin{aligned} S(A_1, B_2, C_3, D_3) = {} & g([v_1[A_1, B_2], C_3], D_3) + g([v_2[A_1, B_2], C_3], D_3) \\ & + g([B_2, v_3[A_1, C_3]], D_3) - g([A_1, v_3[B_2, C_3]], D_3) \\ & + g([[A_1, B_2], C_3], D_3) - g([v_3[A_1, B_2], C_3], D_3) = 0 \end{aligned}$$

by the integrability of $\mathcal{V}_1 \oplus \mathcal{V}_2$ and the Jacobi identity. //

From this lemma and the preceding remarks, we have

THEOREM 10.

$$\mathrm{Pont}^i(\mathcal{V}_2)\,\mathrm{Pont}^j(\mathcal{V}_3) = 0 \text{ for } j > 2p_3 \text{ or } i + j > p_2 + 2p_3.$$

PROOF. If w_2 and w_3 are forms representing elements of $\mathrm{Pont}^i(\mathcal{V}_2)$ and $\mathrm{Pont}^j(\mathcal{V}_3)$, respectively, with $i + j > p_2 + 2p_3$, then $w_2 \wedge w_3$ decomposes as a sum of monomials, each of which vanishes due to the saturation of the exterior products of curvature forms that, according to lemma 9, are not zero. //

Observe that if $p_2 = 0$ (resp. $p_3 = 0$), Bott (resp. Pasternack) theorem is obtained [1] (resp. [7]).

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FINITE GROUPS WHICH ARE AUTOMORPHISM GROUPS OF INFINITE GROUPS ONLY

BY

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ABSTRACT. The object of this paper is to exhibit an infinite set of finite semisimple groups H, each of which is the automorphism group of some infinite group, but of no finite group. We begin the construction by choosing a finite simple group S whose outer automorphism group and Schur multiplier possess certain specified properties. The group H is a certain subgroup of Aut S which contains S. For example, most of the PSL's over a non-prime finite field are candidates for S, and in this case, H is generated by all of the inner, diagonal and graph automorphisms of S.

1. **Introduction**. Our main purpose is the construction of an infinite class of groups which are the automorphism groups of infinite groups, but not of any finite group. The simplest examples of such groups are the quaternion group Q_8, the dicyclic group DC_{12} of order 12 and the binary tetrahedral group BT_{24} of order 24. Each of these groups occurs as the automorphism group of a suitable torsion-free abelian group, as is shown by Hallett and Hirsch (see Fuchs [1], p. 272) in their classification of finite groups which are automorphism groups of torsion-free groups. Straightforward arguments show that none of the groups Q_8, DC_{12}, or BT_{24} is the automorphism group of a finite group. In the first case, this has been pointed out by de Vries and de Miranda [4].

The groups which we construct have a somewhat different structure. Our main result, Theorem A below, has as a consequence that *if* $S = \mathrm{PSL}(7, p^3)$ *where* $p \not\equiv 1 \equiv p^3 \pmod 7$ *and if* H *is the subgroup of* Aut S *generated by the inner, diagonal and graph automorphisms then* (*a*) H *is not the automorphism group of any finite group and* (*b*) H *is the automorphism group of continuously many non-isomorphic countable groups.* Notice that H is isomorphic to the semidirect product $\mathrm{PGL}(7, p^3) \rtimes \langle \tau \rangle$ where τ is the transpose-inverse automorphism.

It is well-known that if $S = \mathrm{PSL}(7, p^3)$ as above, then the outer automorphism group Out S has cyclic Sylow subgroups and the Schur multiplier $M(S)$ is cyclic. Let S be any finite simple group with these properties and let H be a group such that $S \leq H \leq$ Aut S where we identify S and Inn S. We will show first that $H \cong$ Aut G for a finite group G if and only if H/S is self-normalizing in Out S. In addition, certain groups H where H/S is abelian are not automorphism groups of any group. Since Out S is metacyclic, the above facts limit the possibilities for groups H which are automorphism

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groups of infinite groups only. We will prove

THEOREM A. *Let S be a non-abelian finite simple group such that* Out S *has cyclic Sylow subgroups and $M(S)$ is cyclic. Then* Out $S = \langle a\rangle \rtimes \langle b\rangle$, *the semidirect product. If the action of b on $\langle a\rangle$ is neither the identity nor inversion, then there is a subgroup H with* Inn $S < H <$ Aut S *such that*

(a) H is not the automorphism group of any finite group, and

(b) H is the automorphism group of continuously many non-isomorphic countable groups.

Finally, we produce some slightly more complicated examples of automorphism groups of infinite groups only. These are

THEOREM B. *Let p be a prime such that $p = 2q + 1$ where q is an odd prime. Let S be a complete non-abelian finite simple group with $M(S) = 0$. Then there exists a subgroup H of the permutational wreath product S wr* Sym(p) *containing the base group which is the automorphism group of an infinite group but not of any finite group.*

2. **General results**. We will begin by classifying those groups G for which Aut G is finite and semisimple (i.e., it contains no non-trivial abelian normal subgroups). The theorems below are just minor modifications of the work of D. J. S. Robinson [3] and the proofs consist of splitting the theorems into separate cases and then showing that some of them cannot occur.

We will begin by constructing the group $G(Q, F, \epsilon)$ where Q is a finite group with trivial center, F is a torsion-free abelian group and ϵ is an element of Ext (Q_{ab}, F) with the property that $C_{\mathrm{Aut}\,F}(\epsilon) = 1$. Clearly Ext $(Q_{\mathrm{ab}}, F) \cong H^2(Q, F)$ and so we may consider ϵ to be an element of the second cohomology group. Now choose a central extension $F \rightarrowtail G \twoheadrightarrow Q$ with cohomology class ϵ and define $G(Q, F, \epsilon)$ to be G.

Henceforth, we will denote the Aut F-orbit to which ϵ belongs by $\epsilon^{\mathrm{Aut}\,F}$ and the stabilizer in Aut Q of that orbit by $\mathrm{St}_{\mathrm{Aut}\,Q}(\epsilon^{\mathrm{Aut}\,F})$.

THEOREM 2.1. *Let G be a group such that* Aut G *is finite and semisimple. Then there exists $Q \leq$* Aut G *such that*

(a) if G is infinite, then there exists F and ϵ as above such that $G \cong G(Q, F, \epsilon)$, or

(b) if G is finite, then there exists $K \leq M(Q)_{q'}$, where $q = |Q_{\mathrm{ab}}|$ and an elementary abelian 2-group D whose order is not 4 such that $G \cong (\hat{Q}/K) \times D$ where $\hat{Q}$ is the unique stem extension of $M(Q)_{q'}$ by Q. Furthermore, if $D \neq 1$, then $q = |Q_{\mathrm{ab}}|$ and $|M(Q)_{q'} : K|$ are odd.

THEOREM 2.2. *Let Q, F, ϵ, K and D be as in Theorem 2.1. Then*

(i) Aut $(G(Q, F, \epsilon)) \cong \mathrm{St}_{\mathrm{Aut}\,Q}(\epsilon^{\mathrm{Aut}\,F})$ *where $\epsilon^{\mathrm{Aut}\,F}$ is the* Aut F*-orbit to which ϵ belongs.*

(ii) Aut $(\hat{Q}/K \times D) \cong N_{\mathrm{Aut}\,Q}(K) \times$ Aut D.

Moreover, these isomorphisms arise from the obvious induced mappings.

Next, we look at the automorphism group of a semisimple group (i.e., one which has no abelian normal subgroups).

LEMMA 2.3. *Let H be a finite semisimple group with completely reducible radical R. Then* Aut H *is a finite semisimple group with completely reducible radical isomorphic to R. If we identify H and* Aut H *with subgroups of* Aut R *in the natural way, then* Aut $H = N_{\mathrm{Aut}\,R}(H)$.

PROOF. Let $H_0 = \mathrm{Inn}\,H$ and let R_0 be the completely reducible radical of H_0. If A is an abelian normal subgroup of Aut H, then $A \cap H_0 = 1$ and hence $A \leq C_{\mathrm{Aut}\,H}(H_0) = 1$ since $Z(H_0) = 1$. It follows that Aut H is semisimple.

Let S denote the completely reducible radical of Aut H. Clearly $R_0 \leq S$. Since S is completely reducible as an Aut H-operator group and R_0 is Aut H invariant, it follows that $S = R_0 \times R_1$ for some Aut H invariant subgroup R_1. Hence $[H_0, R_1] \leq H_0 \cap R_1 = 1$ and $R_1 \leq C_{\mathrm{Aut}\,H}(H_0) = 1$. Therefore $S = R_0$. We may identify H and Aut H with subgroups of Aut R and the result follows. □

COROLLARY 2.4. *Let H be a finite semisimple group with completely reducible radical R. If H is self-normalizing as a subgroup of* Aut R, *then H is complete and hence it is the automorphism group of a finite group.*

Let H be a finite semisimple group such that $H \cong \mathrm{Aut}\,G$ for some group G. Let R be the completely reducible radical of H. By Theorem 2.1, $G \cong G(Q, F, \epsilon)$ or $G \cong \hat{Q}/K \times D$. Assume that H has no direct factor of type $\mathrm{PSL}(r, 2)$ and so by Theorem 2.2, $H \leq \mathrm{Aut}\,Q$. It follows that $C_H(Q) = 1$. Clearly $Q \cong \mathrm{Inn}\,G$ and so with the natural identifications $Q \leq H \leq \mathrm{Aut}\,R$. Since R is a completely reducible H-operator group and $R \cap Q$ is a normal admissible subgroup, $R = (R \cap Q) \times R_1$ for some H-invariant subgroup R_1. Therefore $R_1 \leq C_H(Q) = 1$ and hence $R \leq Q$. We deduce from this that $\mathrm{Aut}\,Q = N_{\mathrm{Aut}\,R}(Q)$. With the natural identifications, this proves

LEMMA 2.5. *Assume that H is a finite semisimple group with no direct factor of type* $\mathrm{PSL}(r, 2)$. *Let R be the completely reducible radical of H. If $H \cong$* Aut G, *then*

$$R \leq Q = \mathrm{Inn}\,G \leq H \leq \mathrm{Aut}\,Q \leq \mathrm{Aut}\,R.$$

Lemma 2.5 implies that when we are checking if a group is an automorphism group we need only consider normal subgroups Q of H which contain the completely reducible radical R in either $G(Q, F, \epsilon)$ or $\hat{Q}/K \times D$. In order for H to be the automorphism group of an infinite group, there must exist non-trivial Q, F and ϵ which satisfy the necessary conditions for $G(Q, F, \epsilon)$ to exist.

LEMMA 2.6. *Let Q be a finite group with trivial center. Then there exists an infinite group $G(Q, F, \epsilon)$ if and only if Q_{ab} is not an elementary abelian 2-group. In addition, if Q_{ab} is not an elementary abelian 2-group, then there are continuously many non-isomorphic countably infinite groups $G(Q, F, \epsilon)$ with* $|\mathrm{Aut}\,F| = 2$.

PROOF. If Q_{ab} is an elementary abelian 2-group, then the inversion automorphism of F operates trivially on Ext (Q_{ab}, F) and since $C_{\mathrm{Aut}\,F}(\epsilon) = 1$ this implies that F is trivial.

Conversely assume that Q_{ab} has exponent $m > 2$. We will construct continuously many torsion-free abelian groups F_π as follows.

Let π be an infinite set of primes none of which divide m. Define F_π to be the additive group of rationals of the form $t/(p_1 \dots p_k)$ where the p_i are distinct primes contained in π and t is an integer. The only automorphisms of F_π are given by multiplication by 1 or -1.

Writing $Q_{ab} \simeq \operatorname*{Dr}_{i=1,\dots,k} Z_{e_i}$ where e_i divides e_{i+1} and $e_k = m$, we have

$$\operatorname{Ext}(Q_{ab}, F_\pi) \cong \operatorname*{Dr}_{i=1,\dots,k} (F_\pi/e_iF_\pi).$$

Define ϵ_π to be the element of Ext (Q_{ab}, F_π) which corresponds to $(1 + e_1F_\pi, \dots, 1 + e_kF_\pi)$ in the above Aut F_π-isomorphism. Then $C_{\operatorname{Aut} F_\pi}(\epsilon_\pi) = C_{\operatorname{Aut} F_\pi}(1 + mF_\pi) = 1$ since $m > 2$ and Aut F_π is multiplication by either 1 or -1. Hence $G(Q, F_\pi, \epsilon_\pi)$ is countably infinite for each π. ☐

3. **Automorphism groups which are subgroups of Aut S for a simple group S.** Let S be a non-abelian finite simple group whose multiplier $M(S)$ is cyclic and whose outer automorphism group Out S has cyclic Sylow subgroups. We are interested in those subgroups H satisfying $S \leq H \leq \operatorname{Aut} S$. It is clear that there is a correspondence between such groups H and subgroups $\bar{H}$ of Out S by means of the canonical surjection Aut $S \twoheadrightarrow$ Out S. Therefore we will classify all subgroups of Out S.

It is well-known that any finite group G with cyclic Sylow subgroups has presentation

$$(*) \qquad \langle a, b \mid a^m = 1 = b^n, b^{-1}ab = a^r \rangle$$

where $r^n \equiv 1 \pmod m$, m is odd, $1 \leq r < m$ and m and $n(r-1)$ are coprime (Robinson [2], p. 281). If G is non-cyclic, then all subgroups of G are conjugate to a subgroup of the form $\langle a^k, b^\ell \rangle$ where $k | m$ and $\ell | n$. A subgroup K of G is self-normalizing if and only if it is conjugate to a subgroup of the form $\langle a^k, b \rangle$ where $k | m$ or equivalently if and only if n divides $|K|$.

If $H \cong \operatorname{Aut} G$, then any conjugate of H in Aut S is also isomorphic to Aut G. Hence we need only consider when those subgroups H which correspond to subgroups $\bar{H} = \langle a^k, b^\ell \rangle$ of Out $S = \langle a, b \rangle$ are automorphism groups.

Since $\bar{H}$ has cyclic Sylow subgroups, $M(\bar{H}) = 0$. In conjunction with the following lemma this will imply that $M(H)$ is cyclic for any group H satisfying $S \leq H \leq \operatorname{Aut} S$.

LEMMA 3.1. *Let N be a normal subgroup of a group Q such that $M(N)$ is cyclic, N is perfect and $M(Q/N) = 0$. Then $M(Q)$ is cyclic. In addition, if $M(N) = 0$, then $M(Q) = 0$.*

PROOF. If $M(Q) \cong C \leq \tilde{Q}' \cap Z(\tilde{Q})$ where $\tilde{Q}/C \simeq Q$, and $\tilde{N}/C \simeq N$, then $\tilde{N} = \tilde{N}'C$ since N is perfect. Therefore, $\tilde{N}/\tilde{N}' \leq Z(\tilde{Q}/\tilde{N}) \cap (\tilde{Q}/\tilde{N}')'$ and since $(\tilde{Q}/\tilde{N}')/(\tilde{N}/\tilde{N}') \cong Q/N$ and $M(Q/N) = 0$, we have $\tilde{N} = \tilde{N}'$. Then $C \leq \tilde{N}' \cap Z(\tilde{N})$ and since $\tilde{N}/C \cong N$ and $M(N)$ is cyclic $(M(N) = 0)$, $M(Q) \cong C$ is cyclic $(M(Q) \cong C = 0)$ as required. ☐

Let H be a group satisfying $S \leq H \leq \text{Aut } S$ where S is as above and is not of type PSL$(r, 2)$. Assume that $H \cong \text{Aut } G$ for some finite group G. By Theorems 2.1, 2.2 and Lemma 2.5, $H \cong N_{\text{Aut } Q}(K)$ where $S \leq Q \leq H$ and $K \leq M(Q)$. Since $M(Q)$ is cyclic by Lemma 3.1, Aut Q normalizes every subgroup K of $M(Q)$. Hence $N_{\text{Aut } Q}(K) = \text{Aut } Q$. Since Out S has cyclic Sylow subgroups, normality is transitive in Out S (Robinson [2], p. 392). Therefore the subgroup $H = \text{Aut } Q$ is self-normalizing in Aut S. The above argument along with Corollary 2.4 proves

Lemma 3.2. *A group H with the above properties is the automorphism group of a finite group if and only if it is self-normalizing in* Aut S.

Hence the group H of Lemma 3.2 is the automorphism group of a finite group if and only if $\bar{H}$ is conjugate to $\langle a^k, b\rangle$ where $k|m$.

We will now consider groups H where $\bar{H}$ is of the form $\langle a^\ell\rangle$ or $\langle b^k\rangle$ for $\ell \neq 1$ and $k \neq 1$. If H is such a group, then it is not the automorphism group of any group. This follows from

Lemma 3.3. *Let S be a non-abelian finite simple group which is not of type PSL$(r, 2)$ and which has cyclic multiplier. Let H be a subgroup of* Aut S *containing S such that H/S is cyclic. If there exists a subgroup X of* Aut S *properly containing H such that $[H, X] \leq S$, then H is not an automorphism group.*

Proof. Suppose that $H \cong \text{Aut } G$ for some group G. If G is finite, then $H \cong N_{\text{Aut } Q}(K)$ where $S \leq Q \leq H$ and $K \leq M(Q)$ by Theorems 2.1 and 2.2. By Lemma 3.1, $M(Q)$ is cyclic and hence $H \cong N_{\text{Aut } Q}(K) = \text{Aut } Q$. Since $X \leq \text{Aut } Q = H$, we get a contradiction.

If G is infinite, then $H \cong \text{St}_{\text{Aut } Q}(\epsilon^{\text{Aut } F})$ by Theorems 2.1 and 2.2. Since $Q/S = Q_{\text{ab}}$ and $[H, X] \leq S$, we deduce that X operates trivially on Q_{ab} and hence on Ext (Q_{ab}, F). This gives the contradiction that $X \leq H$. Hence H is not an automorphism group. □

One consequence of this lemma is that any non-abelian finite simple group S not of type PSL$(r, 2)$ with $M(S)$ and Out S cyclic has no subgroups H satisfying $S \leq H < \text{Aut } S$ which are automorphism groups.

In order to find automorphism groups of infinite groups only we must concentrate on groups S with Out S metacyclic, but not cyclic. Theorem A will provide us with a group H satisfying $S \leq H < \text{Aut } S$ with $\bar{H} = \langle a, b^k\rangle$ for $k \neq 1$ which is the automorphism group of infinite groups only.

Proof of Theorem A. It is clear that Out S has presentation (*), and so it is a semidirect product.

Let Q be the subgroup of Aut S such that $\bar{Q}$ is $\langle a\rangle$. Since $Q_{\text{ab}} = Q/S$, we have that $m = |Q_{\text{ab}}|$ is odd. By Lemma 2.6, we have continuously many groups $G_\pi = G(Q, F_\pi, \epsilon_\pi)$ with $|\text{Aut } F_\pi| = 2$. Therefore the Aut F_π-orbit of ϵ_π is $\{\epsilon_\pi, -\epsilon_\pi\}$ and by Theorem 2.2, Aut $(G_\pi) \cong \text{St}_{\text{Aut } Q}(\{\epsilon_\pi, -\epsilon_\pi\})$. Since Q is normal in Aut S, it follows from Lemma 2.3 that Aut $(G_\pi) \cong \text{St}_{\text{Aut } S}(\{\epsilon_\pi, -\epsilon_\pi\})$.

Define $H = \mathrm{St}_{\mathrm{Aut}\,S}(\{\epsilon_\pi, -\epsilon_\pi\})$. It is clear that $S \le H \le \mathrm{Aut}\,S$ and that $H \cong \mathrm{Aut}\,(G_\pi)$ for any such set π. Hence H is a semisimple group which is isomorphic to the automorphism group of continuously many non-isomorphic countable groups. Finally, since b acts on $Q_{\mathrm{ab}} = \langle a \rangle$ as multiplication by an integer r not equal to 1 or -1, it cannot stabilize $\{\epsilon_\pi, -\epsilon_\pi\}$. Hence $b \notin \bar{H}$ and $\bar{H} = \langle a, b^{\ell} \rangle$ where $\ell \neq 1$. It follows by lemma 3.2 that H is not the automorphism group of any finite group. ☐

We are now in a position to give some concrete examples.

Theorem 3.4. *Suppose that p and m are primes and r is a positive integer such that $gcd(r, p-1) = 1$, m is odd, and $gcd(r, p^m - 1)$ is a square-free integer greater than one. Then the subgroup H of* Aut (PSL(r, p^m)) *generated by all of the inner, diagonal and graph automorphisms is the automorphism group of continuously many countable groups but no finite groups. Notice that $H \cong \mathrm{PGL}(r, p^m) \rtimes \langle \tau \rangle$ where τ is the transpose-inverse automorphism.*

Proof. Let $S = \mathrm{PSL}(r, p^m)$. The multiplier $M(S)$ is cyclic and $|\mathrm{Out}\ S| = 2m(gcd(r, p^m - 1))$ is square-free. Define the group $\langle a \rangle$ to be the subgroup of all outer diagonal automorphisms and define $\langle b \rangle$ to be the subgroup of outer graph and field automorphisms. It is easy to show that any field automorphism acts in the proper way on $\langle a \rangle$. Hence Theorem A shows the existence of the group H and we may deduce its form from the formula $H = \mathrm{St}_{\mathrm{Aut}\,S}(\{\epsilon, -\epsilon\})$. ☐

In particular, the groups PSL$(7, p^3)$ for $p \not\equiv 1 \pmod 7$ and $p^3 \equiv 1 \pmod 7$ provide an infinite class of examples of automorphism groups of infinite groups only.

4. **More groups which are automorphism groups of infinite groups only**. The techniques used to decide which subgroups of Aut S are automorphism groups can be used to construct more complicated groups which are automorphism groups of infinite groups only.

Proof of Theorem B. Let p be a prime such that $p = 2q + 1$ where q is an odd prime. Let $P = \langle x \rangle$ be a Sylow p-subgroup of the symmetric group Sym (p). Then Aut $P = \langle \sigma \rangle$ where $x^\sigma = x^r$ for some $r > 1$. There exists an element y of Sym (p) of order $(p-1) = 2q$ such that $y^{-1}xy = x^r$. Let $T = \langle x, y \rangle$. Clearly $T = \langle x \rangle \rtimes \langle y \rangle$. Define $M = \langle x, y^q \rangle$ and note that $|y^q| = 2$. Since Sym (p) has $(p-2)!$ Sylow p-subgroups, $|N_{\mathrm{Sym}(p)}(P)| = p(p-1)$ and so $N_{\mathrm{Sym}(p)}(P) = T$. Clearly $N_{\mathrm{Sym}(p)}(M) = T$.

Let S be a complete non-abelian finite simple group with $M(S) = 0$ and define H to be the permutational wreath product $(S \text{ wr } M)$. Then H is a semisimple group with completely reducible radical R equal to the base group. Let Q be any normal subgroup of H containing R. It follows that Q equals either R, $R \rtimes \langle x \rangle$, or H. In each of these cases, $M(Q) = 0$ by Lemma 3.1 and hence if H is the automorphism group of a finite group, then $H = \mathrm{Aut}\ Q$ by Theorems 2.1 and 2.2. Clearly H is properly contained in Aut Q for all possible Q, which shows that H is not the automorphism group of any finite group. Also $H_{\mathrm{ab}} \cong M_{\mathrm{ab}}$ has order 2 and so it follows from Lemma 2.6 that if H is to be an automorphism group then Q cannot equal R or H.

We shall choose Q to be $R \rtimes \langle x \rangle$. Lemma 2.6 gives us an infinite group $G(Q, F, \epsilon)$ with $|\mathrm{Aut}\ F| = 2$ and so

$$\mathrm{Aut}\ (G(Q, F, \epsilon)) \cong \mathrm{St}_{\mathrm{Aut}\, Q}(\{\epsilon, -\epsilon\}).$$

Since S is complete, Aut $Q \cong S$ wr T. A quick calculation shows that $H \cong$ Aut $(G(Q, F, \epsilon))$ and hence H is the automorphism group of an infinite group. ☐

The primes less than 100 to which Theorem B applies are $p = 7, 11, 23, 47, 59$, and 83. It is not known if there are infinitely many such primes.

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FINITE INTERSECTIONS OF PID OR FACTORIAL OVERRINGS

BY

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ABSTRACT. In this paper we study when an integral domain is a finite intersection of PID or factorial overrings. We show that any Krull domain is the intersection of a PID and a field. We give several sufficient conditions for a Krull domain to be an intersection of two PID or factorial overrings.

Introduction. An integral domain R is *locally factorial* if R_x is factorial for each nonzero, nonunit $x \in R$. In [1], we showed that a locally factorial Krull domain is an intersection of two factorial overrings. In particular, a locally factorial Dedekind domain is an intersection of two PID overrings. It is thus natural to ask when an integral domain is a finite intersection of factorial or PID overrings. Such an integral domain is necessarily a Krull domain. If R is a Krull domain, then R is a locally finite intersection of DVR's. Hence a Krull domain is always an intersection of PID overrings. Such a representation gives R as a finite intersection of DVR's if and only if R is a semilocal PID.

In Section 1, we show that if we do not restrict ourselves to overrings, then any Krull domain is an intersection of two PID's. In a like manner, we characterize integrally closed (resp., completely integrally closed) domains as being an intersection of two Bézout (resp., completely integrally closed Bézout) domains. We also show that if R is a Krull domain, then $R[X]$ is an intersection of two PID overrings. In the second section, we use divisor class group techniques to investigate when a Krull domain is a finite intersection of factorial or PID overrings. Our main result, Theorem 2.7, is that a Krull domain with countable divisor class group is an intersection of two factorial overrings. We know of no Krull domain which is not a finite intersection of factorial or PID overrings. We end the paper with several other related open questions.

Any undefined notions and basic facts about Krull domains may be found in [4] or [6]. Throughout, R will always denote an integral domain with quotient field K. Given a Krull domain R, we will denote its divisor class group by $C\ell(R)$, its set of height-one prime ideals by $X^{(1)}(R)$, and the class of a height-one prime ideal P in $C\ell(R)$ by $[P]$. As usual, an *overring* of R is subring of K which contains R. Finally, given $f(X) = a_0 + a_1X + \dots + a_nX^n \in R[X]$, its *content* is the ideal $A_f = (a_0, \dots, a_n)$ of R.

1. **Finite intersections of PID's**. If a DVR V and a field L are both subrings of a larger field F, then it is well-known that $V \cap L$ is again a DVR ([6], Theorem 19.16).

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More generally, suppose that R is a semilocal PID, say $R = V_1 \cap \ldots \cap V_n$, where each V_i is a DVR. Then $R \cap L = (V_1 \cap L) \cap \ldots \cap (V_n \cap L)$ is again a finite intersection of DVR's, and hence is also a semilocal PID. However, if R is a PID and L is a field, then $R \cap L$ need not be a PID, although it is a Krull domain. In fact, we next show that any Krull domain is an intersection of a PID and a field. For the basic facts about Kronecker function rings which we will use in our next several proofs, one may consult ([6], Sections 32, 34, and 44).

THEOREM 1.1. *Let R be an integral domain. Then the following statements are equivalent.*

(1) R is a Krull domain.

(2) R is an intersection of two PID's.

(3) R is an intersection of a PID and a field.

PROOF. Clearly (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (3), we note that the Kronecker function ring of R with respect to the v-operation, $R^v = \{f/g \mid f, g \in R[x], g \neq 0, (A_f)_v \subset (A_g)_v\}$, is a PID ([6], Corollary 44.12) and $R = R^v \cap K$ ([6], Theorem 32.7). □

REMARK 1.2. If R is a Krull domain, then R^v is actually a Euclidean domain ([3], Corollary 5.2 and Theorem 5.3). Also, R^v is a localization of $R[X]$, namely $R^v = R[X]_S$, where $S = \{f \in R[X] \mid (A_f)_v = R\}$ ([5], Theorem 2.5).

In a similar manner, the properties of being integrally closed or completely integrally closed may be characterized in terms of finite intersections of Bézout domains.

THEOREM 1.3. *Let R be an integral domain. Then:*

(1) The following three statements are equivalent.

(a) R is integrally closed.

(b) R is an intersection of two Bézout domains.

(c) R is an intersection of a Bézout domain and a field.

(2) The following three statements are equivalent.

(a) R is completely integrally closed.

(b) R is an intersection of two completely integrally closed Bézout domains.

(c) R is an intersection of a completely integrally closed Bézout domain and a field.

PROOF. (1) This follows as in Theorem 1.1 from the fact that if R is integrally closed, then the Kronecker function ring R^b is a Bézout domain and $R = R^b \cap K$ ([6], Theorem 32.7).

(2) We need only show that if R is completely integrally closed, then R^v is also completely integrally closed. For then, as in (1) above, R^v will be a completely integrally closed Bézout domain and $R = R^v \cap K$. So let $f, g \in R[X]$ with f/g almost integral over R^v. Then there is a nonzero $h \in R[X]$ such that $h(f/g)^n \in R^v$ for each $n \geq 1$. Hence $(A_{hf^n})_v \subset (A_{g^n})_v$. Now $A_h(A_f)_v^n \subset (A_h(A_f)_v^n)_v = (A_{hf^n})_v \subset (A_{g^n})_v = (A_g)_v^n$. Let $0 \neq r \in A_h$. Then $r((A_f)^n((A_g)^{-1})^n)_v \subset R$, so $r(A_f)^n (A_g)^{-n} \subset R$. Let $x \in A_f$ and $y \in (A_g)^{-1}$. Then $r(xy)^n \in R$ for each $n \geq 1$. Hence $xy \in R$ since R is completely

integrally closed. Thus $A_f(A_g)^{-1} \subset R$. Since R is completely integrally closed, the set of v-ideals of R forms a group ([6], Theorem 34.3). Hence $(A_f)_v \subset (A_g)_v$, i.e., $f/g \in R^v$. □

For polynomial rings, the Kronecker function ring techniques yield better results.

THEOREM 1.4. *Let R be an integral domain. Then:*

(1) If R is integrally closed (resp., completely integrally closed), then $R[X]$ is an intersection of two Bézout (resp., completely integrally closed Bézout) domain overrings.

(2) If R is a Krull domain, then $R[X]$ is an intersection of two PID overrings. Moreover, each PID overring may be chosen to be a Euclidean domain which is a localization of $R[X]$.

PROOF. It is sufficient to show that $R[X] = R^* \cap K[X]$ for any e.a.b. $*$-operation on R (see [6], Section 32, for relevant definitions). For if R is integrally closed, then $R[X] = R^b \cap K[X]$; while if R is completely integrally closed, then $R[X] = R^v \cap K[X]$. As in the proof of Theorem 1.3, R^b is a Bézout domain and R^v is a completely integrally closed Bézout domain. If R is a Krull domain, the R^v is a PID. The "moreover" statement follows form Remark 1.2.

Clearly $R[X] \subset R^* \cap K[X]$. Conversely, let $h = c_0 + c_1X + \dots + c_nX^n \in R^* \cap K[X]$. Then $h = (a_0 + a_1X + \dots + a_nX^n)/b$ for some $a_0, a_1, \dots, a_n, b \in R$. Since R^* is well-defined, $(a_0, \dots, a_n) \subset (a_0, \dots, a_n)^* \subset (b)^* = (b)$. Thus each a_i is divisible by b, so $h \in R[X]$. □

It then follows immediately that for a Krull domain R, any localization or subintersection of $R[X]$ is also an intersection of two PID overrings. In particular, this holds for the two rings $R(X)$ and $R\langle X\rangle$. Recall that $R(X) = R[X]_S$ and $R\langle X\rangle = R[X]_U$, where $S = \{f \in R[X] | A_f = R\}$ and $U = \{f \in R[X] | f \text{ is monic}\}$.

REMARK 1.5. If R is a Krull domain, then we have seen that $R[X] = R[X]_S \cap R[X]_T$, where $S = \{f \in R[X] | (A_f)_v = R\}$ and $T = R\backslash\{0\}$ (so $R[X]_T = K[X]$), and each localization is a PID. A natural question is whether this result extends to power series rings. Let R be a Krull domain, $S = \{f \in R[[X]] | (A_f)_v = R\}$, and $T = R\backslash\{0\}$. Then it may be shown that $R[[X]] = R[[X]]_S \cap R[[X]]_T$ and that $R[[X]]_S$ is a PID. However, $R[[X]]_T$ need not be factorial, let alone a PID. For example, if R is factorial but $R[[X]]$ is not factorial (for such a Krull domain, see [4], page 118), then $R[[X]]_T$ can not be factorial by Nagata's Theorem (Theorem 2.1).

2. **Finite intersections of factorial overrings**. In this section, we investigate when a Krull domain R is an intersection of a finite number of factorial subintersections. Let R be a Krull domain with $X = X^{(1)}(R)$ its set of height-one prime ideals. For $Y \subset X$, $R_Y = \bigcap_{P \in Y} R_P$ is also a Krull domain, and is called a *subintersection* of R. Note that $R = R_{Y_1} \cap \dots \cap R_{Y_n}$ if and only if $X = Y_1 \cup \dots \cup Y_n$. This fact follows from the approximation theorem for Krull domains ([4], Theorem 5.8), and it will be used implicitly in the proofs of Theorem 2.5 and Theorem 2.10. Our main tool will be Nagata's Theorem ([4], Theorem 7.1), which relates $C\ell(R)$ to $C\ell(R_Y)$. For future

reference, we include Nagata's Theorem.

THEOREM 2.1. (Nagata's Theorem). *Let R be a Krull domain and R_Y a subintersection of R. Then the natural homomorphism $\phi: C\ell(R) \to C\ell(R_Y)$ is surjective and ker ϕ is generated by $\{[P] | P \in X - Y\}$.*

Recall that each localization R_S is a subintersection; in fact $R_S = R_Y$, where $Y = \{P | P \cap S = \phi\}$. Thus ker $(C\ell(R) \to C\ell(R_S))$ is generated by $\{[P] | P \in X$ and $P \cap S \neq \phi\}$. Also recall that if R is a Dedekind domain, then each overring of R is a subintersection and factorial overrings are just PID's. Our next lemma, an easy consequence of Nagata's Theorem, will prove very useful.

LEMMA 2.2. *Let R be a Krull domain and $P_1, \ldots, P_n \in X = X^{(1)}(R)$. Then $C\ell(R)$ is generated by the classes of $X \backslash \{P_1, \ldots, P_n\}$.*

PROOF. Let $Y = \{P_1, \ldots, P_n\}$. Then $R_Y = R_{P_1} \cap \ldots \cap R_{P_n}$ is a semilocal PID, and hence $C\ell(R_Y) = 0$. By Nagata's Theorem the kernel of $C\ell(R) \to C\ell(R_Y)$, which is $C\ell(R)$, is generated by the classes of $X \backslash Y$. ☐

As mentioned earlier, in ([1], Theorem 2.3 and Proposition 6.1) we showed that if R is a locally factorial Krull domain, then R is an intersection of two factorial overrings. Those proofs actually show that if R is a locally factorial Krull domain, then $R = R_x \cap R_y$ for some nonzero, nonunits $x, y \in R$ with R_x and R_y each factorial. Our next theorem shows to what extent this property characterizes locally factorial Krull domains.

THEOREM 2.3. *Let R be an integral domain. Then the following statements are equivalent.*

(1) R is a Krull domain and $C\ell(R)$ is finitely generated.

(2) There are nonzero $x, y \in R$ such that $R = R_x \cap R_y$ and R_x and R_y are each factorial.

(3) There are nonzero $x_1, \ldots, x_n \in R$ such that $R = R_{x_1} \cap \ldots \cap R_{x_n}$ and each R_{x_i} is factorial.

PROOF. Clearly (2) ⇒ (3). If R_x is factorial for some nonzero $x \in R$, then by Nagata's Theorem $C\ell(R)$ is finitely generated since x is contained in only a finite number of height-one prime ideals. Thus (3) ⇒ (1). So we show that (1) ⇒ (2). We may assume that R is not factorial, and hence R has infinitely many height-one prime ideals. Suppose that $[P_1], \ldots, [P_n]$ generate $C\ell(R)$. Choose $0 \neq x \in P_1 \cap \ldots \cap P_n$. We may assume that these are the only height-one prime ideals that contain x. Since $C\ell(R) = \langle [P] | P \in X^{(1)}(R) \backslash \{P_1, \ldots, P_n\} \rangle$ by Lemma 2.2, there are height-one prime ideals $Q_1, \ldots, Q_m$ of R, distinct from $P_1, \ldots, P_n$, whose classes generate $C\ell(R)$. Choose $y \in (Q_1 \cap \ldots \cap Q_m) \backslash (P_1 \cup \ldots \cup P_n)$. By Nagata's Theorem, R_x and R_y are each factorial. Since x and y belong to no common height-one prime ideals, we have $(x, y)_v = R$. Thus $R = R_x \cap R_y$ ([1], Lemma 2.1). ☐

REMARK 2.4. If R is a Dedekind domain, then $C\ell(R)$ is finitely generated if and only if either (2) or (3) holds with each R_{X_i} a PID. However, one may have $R = R_x \cap R_y$ with R_x and R_y each a PID, but R not a Dedekind domain. For example, let R be a two-dimensional local factorial integral domain.

We next determine conditions on $X^{(1)}(R)$ so that a Krull domain R will be an intersection of a finite number of factorial subintersections.

THEOREM 2.5. *Let R be a Krull domain with $X = X^{(1)}(R)$ and $G = C\ell(R)$. Then the following statements are equivalent.*

(1) $X = X_1 \cup \ldots \cup X_n$, $X_i \cap X_j = \phi$ for $i \neq j$, and $G = \langle [P] | P \in X \backslash X_i \rangle$ for each $i = 1, \ldots, n$.

(2) $X = X_1 \cup \ldots \cup X_n$ and $G = \langle [P] | P \in X \backslash X_i \rangle$ for each $i = 1, \ldots, n$.

(3) $R = R_1 \cap \ldots \cap R_n$, where each R_i, $i = 1, \ldots, n$, is a factorial subintersection of R.

Moreover, if R is a Dedekind domain, then in (3) each R_i is a PID overring of R.

PROOF. Clearly (1) $\Rightarrow$ (2). For (2) $\Rightarrow$ (3), let $R_i = R_{X_i}$ for each $i = 1, \ldots, n$. Then $R = R_1 \cap \ldots \cap R_n$ since $X = X_1 \cup \ldots \cup X_n$. By Nagata's Theorem, each R_i is factorial. Finally, suppose that (3) holds. Since each R_i is a subintersection, $R_i = R_{Y_i}$ for some $Y_i \subset X$. Then $X = Y_1 \cup \ldots \cup Y_n$ since $R = R_1 \cap \ldots \cap R_n$. Define $X_i = Y_i \backslash (X_1 \cup \ldots \cup X_{i-1})$ for each $i = 1, \ldots, n$. Then $X = X_1 \cup \ldots \cup X_n$, and the X_i's are pairwise disjoint. Since each $R_i = R_{Y_i}$ is factorial and $X_i \subset Y_i$, thus $R_{Y_i} \subset R_{X_i}$ and hence each R_{X_i} is factorial. Again, by Nagata's Theorem $G = \langle [P] | P \in X \backslash X_i \rangle$ for each $i = 1, \ldots, n$. ☐

Theorem 2.5 motivates our next observation.

PROPOSITION 2.6. *Suppose that R is an intersection of n factorial or PID subintersections of R. Then any subintersection of R is also an intersection of n such overrings. In particular, if a Dedekind domain R is an intersection of n PID overrings, then any overring of R is also an intersection of n PID overrings.*

PROOF. Suppose that $R = R_1 \cap \ldots \cap R_n$, where each $R_i = R_{Y_i}$ is a factorial subintersection of R. Let R_Y be a subintersection of R. Next, let $Y_i' = Y_i \cap Y$ and $R_i' = R_{Y_i'}$. Then $R_Y = R_1' \cap \ldots \cap R_n'$, and each R_i' is factorial since $Y_i' \subset Y_i$. ☐

If R is a Krull domain with finitely generated divisor class group, then by Theorem 2.3 R is an intersection of two factorial subintersections. Our next theorem extends this to Krull domains with countably generated divisor class groups.

THEOREM 2.7. *Let R be a Krull domain with $G = C\ell(R)$ countable. Then R is an intersection of two factorial subintersections. In addition, if R is a Dedekind domain, then R is an intersection of two PID overrings.*

PROOF. By Theorem 2.3, we may assume that G is infinite. Let $G = \{x_n\}_{n=1}^{\infty}$. Now $x_1 \in \langle [P] | P \in Y_1 \rangle$ for some finite $Y_1 \subset X = X^{(1)}(R)$. By Lemma 2.2, also $x_1 \in \langle [P] | P \in Z_1 \rangle$ for some finite $Z_1 \subset X$ with $Y_1 \cap Z_1 = \phi$. Continuing this process, for each positive integer n there are disjoint finite subsets Y_n and Z_n of X, each pairwise disjoint from the previously defined Y_i's and Z_i's, such that $x_n \in \langle [P] | P \in Y_n \rangle$ and $x_n \in \langle [P] | P \in Z_n \rangle$. Let $Y = \bigcup_{n=1}^{\infty} Y_n$ and $Z = \bigcup_{n=1}^{\infty} Z_n$. Then $Y \cap Z = \phi$ and $G = \langle [P] | P \in Y \rangle = \langle [P] | P \in Z \rangle$. Next, let $X_1 = Y$ and $X_2 = X \backslash X_1 \supset Z$. Then $X = X_1 \cup X_2$, $X_1 \cap X_2 = \phi$, and $G = \langle [P] | P \in X_i \rangle$ for $i = 1, 2$. Hence by Theorem 2.5, R is an intersection of two factorial subintersections. ☐

REMARK 2.8. In particular, Theorem 2.7 is applicable if either R or $X^{(1)}(R)$ is countable. We know of no Krull domain which is not an intersection of two factorial (in fact, PID) subintersections. Thus the countable version of Claborn's construction of Dedekind domains ([2], Theorem 2.1, or [4], Theorem 15.18) can not be used to obtain a Dedekind domain which is not an intersection of two PID overrings.

Our next example shows that $C\ell(R)$ alone cannot be used to determine if there are any Krull domains which are not a finite intersection of PID overrings.

EXAMPLE 2.9. *Let G be any abelian group. Then for any positive integer n (or ∞), there is a Krull domain R of dimension n with $C\ell(R) = G$ which is an intersection of two PID overrings, each of which is a localization of R. We do the case when $n = 1$ (i.e., R is a Dedekind domain); the case when $n > 1$ then follows easily from Theorem 1.4. By Claborn's Theorem ([4], Theorem 14.10) there is a Dedekind domain A with $C\ell(A) = G$. Then $R = A\langle X \rangle$ is a Dedekind domain ([8], Proposition 2.3), and it may be shown that $C\ell(R) = G$. By the remark after Theorem 1.4, R is an intersection of two PID overrings, each of which is a localization of R. Alternately, ([7], Theorem 2.3) may be used to construct a Dedekind domain R with $C\ell(R) = G$ which is an intersection of two PID overrings.*

Our final two results may be viewed as companion theorems to Theorems 2.3 and 2.7.

THEOREM 2.10. *Let R be a Krull domain (resp., Dedekind domain) with $G = C\ell(R)$. Then the following statements are equivalent.*

(1) G is finitely generated.

(2) $R = R_1 \cap R_2$, where R_1 and R_2 are subintersections with R_1 factorial (resp., a PID) and R_2 a semilocal PID.

PROOF. (1) $\Rightarrow$ (2). Say that $[P_1], \ldots, [P_n]$ generate G. Let $R_2 = R_{P_1} \cap \ldots \cap R_{P_n}$ and $R_1 = R_Y$ for $Y = X^{(1)}(R) \backslash \{P_1, \ldots, P_n\}$. Clearly $R = R_1 \cap R_2$ and R_2 is a semilocal PID. Also, by Nagata's Theorem R_1 is factorial. (2) $\Rightarrow$ (1). Let $R_2 = R_{P_1} \cap \ldots \cap R_{P_n}$ and $R_1 = R_Z$ for some $P_1, \ldots, P_n \in X = X^{(1)}(R)$ and $Z \subset X$. Also, let $Y = X \backslash \{P_1, \ldots, P_n\}$. Then $Y \subset Z$, and thus R_Y is factorial. Hence $G = \langle [P_1], \ldots, [P_n] \rangle$ by Nagata's Theorem again. ☐

THEOREM 2.11. *Let R be a two-dimensional Krull domain with only a finite number of height-two maximal ideals, and let $G = C\ell(R)$. Then:*

(1) If G is finitely generated, then R is an intersection of a PID overring and a semilocal PID overring.

(2) If G is countable, then R is an intersection of two PID overrings and a semilocal PID overring.

PROOF. Let $M_1, \dots, M_n$ be the height-two maximal ideals of R. Choose $0 \neq x \in M_1 \cap \dots \cap M_n$. Let $P_1, \dots, P_k$ be the height-one prime ideals of R which contain x. Then $R = R_x \cap R_{P_1} \cap \dots \cap R_{P_k}$. Now R_x is a Dedekind domain and $R_{P_1} \cap \dots \cap R_{P_k}$ is a semilocal PID. Parts (1) and (2) now follow from Theorems 2.10 and 2.7, respectively. ☐

We close with several open questions.

QUESTION 1. Is each integrally closed (resp., completely integrally closed) domain an intersection of two Bézout (resp., completely integrally closed Bézout) domain overrings?

QUESTION 2. Is each Krull domain an intersection of a finite number of PID or factorial overrings?

QUESTION 3. If R is a finite intersection of PID or factorial overrings, is R actually an intersection of two such overrings?

There are also several other variants of the above questions. For example, we may ask if the overrings may be chosen to be localizations, subintersections, or flat overrings of R.

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INTEGRALLY CLOSED CONDENSED DOMAINS ARE BÉZOUT

BY

DAVID F. ANDERSON, JIMMY T. ARNOLD AND DAVID E. DOBBS

ABSTRACT. It is proved that an integral domain R is a Bézout domain if (and only if) R is integrally closed and $IJ = \{ij \mid i \in I, j \in J\}$ for all ideals I and J of R; that is, if (and only if) R is an integrally closed condensed domain. The article then introduces a weakening of the "condensed" concept which, in the context of the $k + M$ construction, is equivalent to a certain field-theoretic condition. Finally, the field extensions satisfying this condition are classified.

1. **Introduction**. Condensed (commutative integral) domains were introduced by Anderson and Dobbs in [1]. Recall that a domain R is *condensed* if $IJ = \{ij \mid i \in I, j \in J\}$ for all ideals I and J of R. Bézout domains provide the most obvious class of condensed domains, but as ([1], Example 2.3) illustrates, a condensed domain need not be a Bézout domain. This article is motivated by the results in [1] that characterize condensed domains within certain larger classes of domains. Specifically, a GCD-domain is condensed if and only if it is a Bézout domain ([1], Proposition 2.12); a Prüfer domain is condensed if and only if it is a Bézout domain ([1], Corollary 2.6); and a Krull domain is condensed if and only if it is a PID ([1], Proposition 2.13). Since GCD-domains, Prüfer domains, and Krull domains are integrally closed, these results are immediate consequences of our

MAIN THEOREM. *An integral domain is a Bézout domain if (and only if) it is integrally closed and condensed.*

Seeking to sharpen the above characterization of Bézout domains, we introduce in Section 3 the notion of a quasicondensed domain. The precise definition of this weakening of the "condensed" concept (given below) is motivated by the nature of the proofs and examples in [1] and Section 2. It will be shown that there exists an integrally closed quasicondensed domain which is not a Bézout domain. More generally, the question of which D + M constructions lead to quasicondensed domains is reduced in Theorem 4 to the problem of characterizing a certain type of field extension. The underlying field theory is then treated in Theorem 5.

Throughout, R denotes a domain. Any unexplained terminology is standard, as in [2] and [5].

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2. **Proof of the main theorem**. We need only prove that any integrally closed condensed domain R is a Bézout domain. By ([1], Corollary 2.6), it suffices to show that R is a Prüfer domain. We do this by showing that R_M is a valuation domain for each maximal ideal M of R ([5], Theorem 64). By ([1], Proposition 2.4), each overring of a condensed domain is again a condensed domain. Thus each R_M is a condensed domain, and we have reduced to showing that any integrally closed quasilocal condensed domain is a valuation domain.

Let R be such an integral domain, and let z be a nonzero element in the quotient field of R. We must show that either z or z^{-1} is in R. Write $z = xy^{-1}$ where $x, y \in R$, and then set $I = (x^2, y^2)$ and $J = (x, y)$. Since $x^3 + y^3 \in IJ$ and R is condensed, there exist a, b, c, d in R such that

$$x^3 + y^3 = (ax^2 + by^2)(cx + dy) = acx^3 + adx^2y + bcxy^2 + bdy^3.$$

Collecting terms and dividing by y^3 yields

$$(1 - ac)z^3 - adz^2 - bcz + (1 - bd) = 0.$$

Since R is quasilocal, it is readily seen that at least one of the coefficients of this polynomial is a unit. Thus by the u, u^{-1}-Lemma ([5], Theorem 67), either $z \in R$ or $z^{-1} \in R$. □

3. **Quasicondensed domains**. We shall say that an integral domain R is *quasicondensed* if $I^n = \{i_1 \dots i_n | i_j \in I \text{ for } 1 \leq j \leq n\}$ for each positive integer n and each two-generated ideal $I = (a, b)$ of R. It is clear that each condensed domain is quasicondensed. However, an integrally closed quasicondensed domain need not be condensed. According to the Main Theorem, this last statement means that an integrally closed quasicondensed domain need not be a Bézout domain. An example which illustrates this is $R = \mathbb{C} + X\mathbb{C}(Y)[[X]]$. A major portion of this section will be devoted to establishing (a generalization of) this assertion. First, we find consequences of "quasicondensed" in two classical settings.

PROPOSITION 1. *If R is a Noetherian quasicondensed domain, then depth $(R) \leq 1$.*

PROOF. This follows from the proof of ([1], Theorem 3.3) using the easily proved fact that any localization of a quasicondensed domain is again quasicondensed. □

PROPOSITION 2. *Let (R, M) be an integrally closed quasilocal quasicondensed domain, with residue field k $(=R/M)$. Then either R is a valuation domain, or k is algebraically closed.*

PROOF. Suppose that R is not a valuation domain. Choose $a, b \in R$ such that $I = (a, b)$ is not principal. For $d \in R$, let $\bar{d}$ denote its image in k. We shall show that any $f \in k[X]$ of degree $n \geq 1$ has a root in k. Write $f(X) = \sum_{j=0}^{n} \bar{\alpha}_j X^j \in k[X]$, for suitable $\alpha_j \in R$. As $\sum_{j=0}^{n} \alpha_j a^j b^{n-j} \in I^n$, the fact that R is quasicondensed allows us to write $\sum_{j=0}^{n} \alpha_j a^j b^{n-j} = (\beta_0 b + \beta_1 a)(\sum_{j=0}^{n-1} \gamma_j a^j b^{n-1-j})$ for suitable $\beta_j, \gamma_j \in R$. After multiplying and collecting terms, and then dividing by b^n, we find $\sum_{j=0}^{n} \delta_j (ab^{-1})^j = 0$, where $\delta_0 = \alpha_0 - \beta_0\gamma_0$, $\delta_n = \alpha_n - \beta_1\gamma_{n-1}$, and $\delta_j = \alpha_j - \beta_1\gamma_{j-1} - \beta_0\gamma_j$, for $1 \leq j \leq n - 1$. Since neither ab^{-1} nor ba^{-1} is in R, it follows from the u, u^{-1}-Lemma ([5], Theorem 67) that $\delta_j \in M$ for each j. Thus applying $(\bar{\ })$ yields $\bar{\alpha}_0 = \bar{\beta}_0\bar{\gamma}_0$, $\bar{\alpha}_j = \bar{\beta}_1\bar{\gamma}_{j-1} + \bar{\beta}_0\bar{\gamma}_j$ for $1 \leq j \leq n - 1$, and $\bar{\alpha}_n = \bar{\beta}_1\bar{\gamma}_{n-1}$. Accordingly, one has the factorization $f(X) = (\bar{\beta}_1 X + \bar{\beta}_0)(\bar{\gamma}_{n-1}X^{n-1} + \dots \bar{\gamma}_0)$ in $k[X]$, and the result follows. □

We next assemble material that will lead to a partial converse of Proposition 2.

PROPOSITION 3. *Let K/k be a field extension. Let $T = K + M$ be a quasi-local domain with maximal ideal M. Let R denote the (quasilocal) domain $k + M$. Let $I = (a, b)$ be a proper ideal of R, with $ba^{-1} = \alpha + m \in T$ ($\alpha \in K$, $m \in M$). Set $W = k + k\alpha$, viewed as a k-subspace of K. Then:*

(a) $I^n = W^n a^n + Ma^n$ for all $n \geq 1$.

(b) For $n \geq 1$, $I^n = \{i_1 \dots i_n \mid \text{each } i_j \in I\}$ if and only if $\sum_{i=0}^{n} k\alpha^i = \prod_{i=1}^{n} (k + k\alpha)$.

PROOF. The proof of (*a*) is straightforward (cf. [3], Lemma 3.2).

(*b*) ($\Rightarrow$): One inclusion holds in general. Conversely, let $\beta \in \sum_{i=0}^{n} k\alpha^i$. By (*a*), $\beta a^n \in I^n$, and so the hypothesized description of I^n yields $\beta a^n = (\beta_1 a + m_1 a) \dots (\beta_n a + m_n a)$ for some $\beta_i \in k + k\alpha = W$ and some $m_i \in M$. Since $W^n a^n \cap Ma^n = 0$, $\beta = \beta_1 \dots \beta_n \in \prod_{i=1}^{n} (k + k\alpha)$.

($\Leftarrow$): Let $x \in I^n \backslash \{0\}$. By (a), write $x = \beta a^n + ma^n$, for some $\beta \in W^n$, $m \in M$. We may assume that $\beta \neq 0$. Since $\beta = \beta_1 \dots \beta_n$ for some nonzero elements $\beta_i \in W$, $x = (\beta_1 a) \dots (\beta_{n-1} a)(\beta_n a + \gamma m a)$, where $\gamma = (\beta_1 \dots \beta_{n-1})^{-1}$. In particular, $x \in \{i_1 \dots i_n \mid \text{each } i_j \in I\}$. □

In view of Proposition 3, the next definition seems timely. An extension K/k of fields is said to satisfy property (*) if $\sum_{i=0}^{n} k\alpha^i = \prod_{i=1}^{n} (k + k\alpha)$ for each $\alpha \in K$ and $n \geq 1$.

THEOREM 4. *Let V be a valuation domain of the form $K + M$, where K is a field and M ($\neq 0$) is the maximal ideal of V. Let k be a subfield of K, and set $R = k + M$. Then R is quasicondensed if and only if K/k satisfies (*).*

PROOF. This follows easily from Proposition 3(*b*) in view of the following two observations. First, for nonzero a, $b \in R$, either ab^{-1} or ba^{-1} is in V since V is a valuation domain. Secondly, since M is nonzero, each nonzero $\alpha \in K$ may be written as $\alpha = ba^{-1}$ for some nonzero a, $b \in M$. (Choose $0 \neq m \in M$; let $a = m$ and $b = \alpha m$.) □

Thus, we have reduced the question of which $k + M$'s are quasicondensed to the question of which field extensions K/k satisfy (*). Here is the answer to the latter question.

THEOREM 5. *An extension K/k of fields satisfies (*) if and only if either*

(a) k is algebraically closed, or

(b) K/k is algebraic and for each $\alpha \in K$, each nonconstant $f \in k[X]$ with $\deg f < [k(\alpha):k]$ splits over k.

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PROOF. If $K = k$, then $(*)$ and (b) hold, so we may assume that $K \neq k$. Choose $\alpha \in K\backslash k$. First, suppose that (a) holds. Then any $\sum_{i=0}^{n} a_iX^i \in k[X]$ factors as $b\prod_{i=1}^{n}(X - c_i)$ for some $b, c_i \in k$. Hence $\sum_{i=0}^{n} a_i\alpha^i = b\prod_{i=1}^{n}(\alpha - c_i)$. Next, suppose that (b) holds. Let $m = \deg(\alpha)$. As above, we then have $\sum_{i=0}^{n} k\alpha^i = \prod_{i=1}^{n}(k + k\alpha)$ for each $n < m$. For $n \geq m$, just note that $\sum_{i=0}^{n} k\alpha^i = \sum_{i=0}^{m-1} k\alpha^i$ and $\prod_{i=1}^{n}(k + k\alpha) \supset \prod_{i=1}^{m-1}(k + k\alpha)$ since $\deg(\alpha) = m$.

Conversely, suppose that K/k satisfies $(*)$. If K/k is not algebraic, select $\alpha \in K$ transcendental over k; interpreting $(*)$ in $k[X]$ $(\cong k[\alpha])$, we see that k is algebraically closed. Thus we may assume that K/k is algebraic. Let $\alpha \in K\backslash k$, with g the minimal polynomial of α over k; set $n = \deg g$. Let $f(X) = a_0 + a_1X + \ldots + a_mX^m \in k[X]$, with $1 \leq m < n$. Since K/k satisfies $(*)$,

$$a_0 + a_1\alpha + \ldots + a_m\alpha^m = (b_1 + c_1\alpha)\ldots(b_m + c_m\alpha)$$

for some $b_i, c_i \in k$. Let $h(X) = (b_1 + c_1X)\ldots(b_m + c_mX)$. Then $f(X) - h(X)$ has α as a root, and so is divisible by g. Since $m < n$, necessarily $f(X) - h(X) = 0$, so $f = h$ splits over k. ☐

Combining Theorems 4 and 5 (with the Fundamental Theorem of Algebra), we see that $R = \mathbb{C} + X\mathbb{C}(Y)[[X]]$ is quasicondensed. It is well known that R is quasilocal and integrally closed, but not a Bézout domain (cf. [2], Exercise 11, page 202 and Exercise 13, page 286). While each localization of a quasicondensed domain is quasicondensed, an overring of quasicondensed domain need not be quasicondensed. (Contrast with the "condensed" case [1], Proposition 2.4.) Indeed, by Theorems 4 and 5, the overring $\mathbb{C}(Y^3) + X\mathbb{C}(Y)[[X]]$ is not quasicondensed.

If K/k is a field extension such that each $\alpha \in K$ has $[k(\alpha):k] \leq 2$, then clearly K/k satisfies $(*)$. In particular, if $[K:k] = 2$, then K/k satisfies $(*)$. The next remark investigates $(*)$ in greater detail.

REMARK 6. (a) If an algebraic field extension K/k satisfies $(*)$, then K/k is either separable or purely inseparable. To show this, we may assume that $\operatorname{char}(k) = p > 0$. If $\alpha \in K$ is not purely inseparable over k, then $k(\alpha^{p^n}) = k(\alpha)$ for each $n \geq 1$ since K/k satisfies $(*)$. Hence each element of K is either separable or purely inseparable over k ([4], Exercise 9, page 288). By [4], Exercise 4, page 288, the field extension K/k is either separable or purely inseparable.

(b) Suppose that k is not algebraically closed and that K/k is separable and satisfies $(*)$. Then K/k is finite. For if K/k is infinite, then K has elements of arbitrarily high degree ([6], Lemma 1, page 194), contradicting $(*)$. If K/k is a proper finite Galois field extension which satisfies $(*)$, then $[K:k]$ is a prime. This follows easily from the Fundamental Theorem of Galois Theory and the Primitive Element Theorem.

Let K/k be a purely inseparable field extension with $\operatorname{char}(k) = p$ which satisfies $(*)$. Then $[k(\alpha):k] = p$ for each $\alpha \in K\backslash k$. However, unlike the separable case, such a purely inseparable field extension can be infinite dimensional. For example, let $k = \mathbb{Z}/2\mathbb{Z}(\{X_n^2\}_{n=1}^{\infty})$ and $K = \mathbb{Z}/2\mathbb{Z}(\{X_n\}_{n=1}^{\infty})$. Then K/k is an infinite purely inseparable

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field extension which satisfies (*) since each $\alpha \in K\backslash k$ has degree two over k.

(c) We show that for each prime p there is a Galois field extension K/k of degree p which satisfies (*). Our example is modeled after Artin's proof (cf. [6], Theorem 1, page 169) of the existence of algebraic closures. Let $k_0 = Q$. For each $n \geq 0$, we define inductively a field extension k_n. If k_n has been defined, then k_{n+1} is the field generated over k_n by all elements of $\mathbb{C}$ which have degree strictly less than p over k_n. Then let $k = \bigcup_{n=1}^{\infty} k_n$ and $K = k(2^{1/p})$. Since the degree over Q of each $\alpha \in k$ is not divisible by p, $[K:k] = p$. By construction k contains all pth roots of unity; thus K is a splitting field for $X^p - 2$ over k and hence is Galois. Finally we show that K/k satisfies (*). Let $f \in k[X]$ have $1 \leq \deg f < p$. Then $f \in k_n[X]$ for some n, and hence by construction f splits over $k_{n+1} \subset k$.

By ([1], Proposition 2.5), each condensed domain has trivial Picard group, i.e., each invertible ideal is principal. As a closing question, we ask if, analogously, each quasicondensed domain also has trivial Picard group. If this were true, then by Proposition 1, an integrally closed quasicondensed Noetherian domain would be a PID. In closing, we do have the following partial result.

PROPOSITION 7. *Let I be an ideal of a domain R with $I^n = xR$ for some $n \geq 1$ and $x \in R$. If $I^n = \{i_1 \ldots i_n \mid \text{each } i_j \in I\}$, then I is principal.*

PROOF. By hypothesis, $x = i_1 \ldots i_n$ for some elements $i_j \in I$. Since each $(i_j) \subset I$ and $I^n = xR = (i_1) \ldots (i_n) \subset I^n$, it readily follows that $I = (i_j)$ for each $1 \leq j \leq n$. ☐

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ANOTHER CLASS OF CYCLICLY EXTENSIBLE AND REDUCIBLE PROPERTIES

BY

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ABSTRACT. A space S has property P_{-1} if S is nonempty. For $n > -1$, S has property P_n if it is locally connected, has property P_{n-1} and if whenever it is written as a union, $S = A \cup B$ where each of A and B is closed and has property P_{n-1}, then $A \cap B$ also has property P_{n-1}. The purpose of this paper is to establish that for locally compact spaces, each of the properties P_n is both cyclicly extensible and reducible.

Introduction. In [3], it was shown that k-coherence is, under certain conditions, both cyclicly extensible and reducible. In this paper, we define a weaker class of properties similar to properties defined by Vietoris in 1932 [14]. We show that for all connected, locally connected and locally compact Hausdorff spaces, these properties are all cyclicly extensible and reducible.

1. PRELIMINARIES. Throughout this paper, M denotes a connected and locally connected Hausdorff space.

A point p of M is an *end point* of M if p has a neighbourhood base of open sets having singleton boundaries. p is a *cut* point of M if $M - p$ is disconnected. Two points a and b of M are said to be *conjugate in* M if and only if no point of M separates a and b in M. $E(a,b)$ denotes the collection of all points of M which separate a and b in M and "<" denotes the *cut point order* on $E(a,b) \cup \{a,b\}$ – i.e. $a < x$, $x < b$ for all $x \in E(a,b)$; $a < b$, and if $x, y \in E(a,b)$ then $x < y$ if and only if $x \in E(a,y)$. (It has been established, ([15 and [2]), that $E(a,b) \cup \{a,b\}$ is closed and compact and that the subspace topology on $E(a,b) \cup \{a,b\}$ is the order topology relative to the cut point order.) An A-set of M is a closed subset of M such that every component of $M - A$ has singleton boundary. If a, $b \in M$, then $C(a,b) = \cap \{A\colon A \text{ is an } A\text{-set of } M \text{ and } a, b \in A\}$ and is called the *cyclic chain in M from a to b*. If $E \subset M$, then E is an E_0-set of M if E is nondegenerate, connected, contains no cut point of itself and is maximal with respect to these properties. A *cyclic element* of M is a subset of M which is an E_0-set of M or is a singleton cut point or end point of M. A property is *cyclicly reducible* if whenever a space M has the property, then every cyclic element of M has the property; a property is *cyclicly extensible* if whenever every cyclic element of M has the property, then M does also.

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2. Definition. *A space has property P_{-1} if S is non-empty. S has property P_n, $n > -1$ if S is locally connected, has property P_{n-1} and whenever $S = A \cup B$ is a division of S into closed sets each having property P_{n-1}, then $A \cap B$ also has property P_{n-1}. We note that P_0 is connectedness plus local connectedness and P_1 is unicoherence plus local connectedness.*

3. Notation. In the rest of this paper, if $S \subset A$ for some A-set A, then S^* denotes the union of S with all components C of $M - A$ such that the boundary of C belongs to S.

4. Lemma. *If M is locally compact, then for each nonnegative integer n, if M has property P_{n-1}, then M has property P_n if and only if every cyclic element of M has property P_n.*

Proof. We state an induction hypothesis $I(n)$: If M is locally compact, then

(a) if M has property P_n, then every cyclic element of M has property P_n;

(b) if M has property P_{n-1} and every cyclic element of M has property P_n; then M has property P_n;

(c) if M has property P_n, then every A-set of M has property P_n; and if A is an A-set of M and Z is any locally compact subset of M such that $A \cap Z \neq \phi$ and Z has property P_n, then $A \cap Z$ has property P_n;

(d) if M has property P_n, A is an A-set of M, and $A = S \cup T$ is a division of M into closed sets each having property P_n, and if $L = S^*$ and $N = T \cup (A - S)^*$, then $M = L \cup N$ is a division of M into closed sets each having property P_n.

$I(0) - a$ and b are immediate and $I(0) - c$ and d have been established elsewhere, ([5], Theorem 5.3 and [3], Theorem 3, respectively).

Assume that $I(n - 1)$ has been established, $n \geq 1$. Suppose that M has property P_n and that E is an E_0-set of M. Then M has property P_{n-1}, so by $I(n - 1)$, E has property P_{n-1}. Suppose $E = S \cup T$ is a division of E into closed sets each having property P_{n-1}. Let $L = S^*$, $N = T \cup (E - S)^*$. By $I(n - 1)$, $M = L \cup N$ is a division of M into closed sets each having property P_{n-1}. Since M has property P_n, $L \cap N$ has property P_{n-1}. Further, it was established in [3], (Theorem 3) that $L \cap N = S \cap T$. Thus E has property P_n, and it follows that every cyclic element of M has property P_n. Thus $I(n - 1)$ implies $I(n) - a$.

Now assume that M has property P_{n-1} and that every cyclic element of M has property P_n. Suppose $M = S \cup T$ is a division of M into closed sets each having property P_{n-1}. Then each of S and T is connected and locally connected and $S \cap T$ is locally compact. We now show that $S \cap T$ is connected and locally connected.

Suppose that $S \cap T = W \cup Z$ where W and Z are disjoint closed sets. If E is an E_0-set of M, then $E = (E \cap S) \cup (E \cap T)$ and since S and T are each connected, it follows from Theorem 5.3 of [6] that $E \cap S$ and $E \cap T$ are each connected and locally connected. Since E has property P_n, $E \cap S \cap T$ is connected. Thus $E \cap S \cap T \subset W$ or $E \cap S \cap T \subset Z$. Let $w \in W$, $z \in Z$. Since no E_0-set of M meets both W and Z, $E(w, z) \neq \phi$. Since w, $z \in S \cap T$, and S and T are each connected, $E(w, z) \subset S \cap$

T. Let t_1 be the last point in $(E(w,z) \cup \{w,z\}) \cap W$ and t_2 be the first point in $(E(w,z) \cup \{w,z\}) \cap Z$. Then $t_1 \neq t_2$ and t_1 and t_2 are conjugate in M. It follows ([5], Theorem 6.2) that $C(t_1, t_2)$ is an E_0-set of M that meets both W and Z. Since this is a contradiction, $S \cap T$ is connected.

Suppose now that $S \cap T$ is not locally connected. Then there is an open set of M and a point $p \in S \cap T$ such that p lies on a continuum of convergence $D = \lim_{\alpha} D_\alpha$, where for each α, D_α is *the closure* of a component C_α of $O \cap S \cap T$ and the components of $O \cap S \cap T$ containing D and D_α are distinct, ([1], Theorem 8). Since D is a continuum of convergence of M, there is an E_0-set E of M such that $D = \lim_{\alpha} E \cap D_\alpha$ ([2], Theorem 5.13). This implies that $E \cap S \cap T$ is nondegenerate. Now $E \cap S$ and $E \cap T$ are connected and locally connected and for each α, $E \cap D_\alpha$ is contained in a component of $O \cap S \cap T \cap E$ distinct from the component of $O \cap S \cap T \cap E$ containing $E \cap D$. But this implies that $E \cap S \cap T$ is not locally connected, which is a contradiction since E has property P_n. Thus $S \cap T$ is locally connected.

Now let E be an E_0-set of $S \cap T$. $E \subset \tilde{E}$ for some E_0-set $\tilde{E}$ of M. Since M has property P_{n-1}, $I(n-1)$ implies that each of $\tilde{E}$, $S \cap \tilde{E}$ and $T \cap \tilde{E}$ has property P_{n-1}. Further since $\tilde{E}$ has property P_n, $\tilde{E} \cap S \cap T$ has property P_{n-1}. Now E is an E_0-set of $\tilde{E} \cap S \cap T$, so by $I(n-1)$, E has property P_{n-1}. It follows that evey cyclic element of $S \cap T$ has property P_{n-1}, so again by the induction hypothesis, $S \cap T$ has property P_{n-1}. Thus M has property P_n and $I(n) - b$ is established.

Now assume that M has property P_n and that A is an A-set of M. By $I(n-1)$, A has property P_{n-1} (since M does). If E is a cyclic element of A, then E is a cyclic element of M. It follows, since $I(n-1)$ implies $I(n) - a$, that E has property P_n. Thus every cyclic element of A as property P_n and since $I(n-1)$ implies $I(n-1) - b$, A has property P_n. Further, if Z is any locally compact subset of M such that $A \cap Z \neq \phi$ and Z has property P_n, then since $A \cap Z$ is an A-set of Z, it follows from what we have just proved that $A \cap Z$ has property P_n. Thus $I(n) - c$ is proved.

Finally, assume that M has property P_n, that A is an A-set of M and that $A = S \cup T$ is a division of A into closed sets each having property P_n. Let $L = S^*$, $N = T \cup (A - S)^*$. By $I(n-1)$, L and N each have property P_{n-1}. Let E be an E_0-set of L. Then either $E \subset A$ or $E \subset \bar{C}$ for some component C of $M - A$ such that $C \subset L$. If the latter, then E is an E_0-set of $\bar{C}$ and therefore of M, and it follows from what has already been proved that E has property P_n. If $E \subset A$, let $\tilde{E}$ be the E_0-set of M such that $E \subset \tilde{E}$. Then $\tilde{E}$ has property P_n. Further, since S is a locally compact subset of M and has property P_n, it again follows from what has already been proved that $S \cap \tilde{E}$ has property P_n. Since E is an E_0-set of $S \cap E$, E has property P_n. Thus every cyclic element of L has property P_n, so L has property P_n. The case for N is similar, so $I(n) - d$ is established and the Lemma is proved.

The following corollary is immediate.

5. Corollary. *If M is locally compact, then for every nonnegative integer n, if M has property P_n, and A is an A-set of M, then*

(a) A has property P_n;

(b) if Z is a locally compact subset of M that meets A and has property P_n, then $A \cap Z$ has property P_n;

(c) if $A = S \cup T$ is a division of A into closed sets each having property P_n and $L = S^*$, $N = T \cup (A - S)^*$, then $M = L \cup N$ is a division of M into closed sets having property P_n.

6. THEOREM. *If M is locally compact and n is an integer ≥ -1, then M has property P_n if and only if every cyclic element of M has property P_n.*

PROOF. If $n = -1$ or $n = 0$, the result is immediate, so we assume $n > 0$. If M has property P_n, then from our Lemma, every cyclic element of M has property P_n. Suppose that every cyclic element of M has property P_n and that M does not. Since M is connected and locally connected, M has property P_0. Let n^* be the first integer ($n^* > 0$) such that M does not have property P_{n^*}. Then $0 < n^* \leq n$ and M has property P_{n^*-1}. Since every cyclic element of M has property P_n, evey cyclic element of M has property P_{n^*} so from our Lemma, M has property P_{n^*}, contrary to the definition of n^*. The theorem follows.

The following is trivial.

7. COROLLARY. *Every dendron has property P_n for every nonnegative integer n.*

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ON QUASISIMILARITY FOR TOEPLITZ OPERATORS

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ABSTRACT. In this article we give a sufficient condition for quasisimilar analytic Toeplitz operators to be unitarily equivalent. We also use a result of Deddens and Wong to give a sufficient condition for an operator intertwining two analytic Toeplitz operators to intertwine their inner parts too. Analytic Toeplitz operators with univalent symbols satisfying a suitable normalization that are quasisimilar are shown to have equal symbols.

1. **Introduction**. Let H^2 denote the Hilbert space of functions f analytic in the open unit disk $\mathbb{D}$ which satisfy $\sup_{0\leq r<1} \int |f(re^{i\theta})|^2\, d\theta < \infty$. Let H^∞ be the space of bounded analytic functions on $\mathbb{D}$, and for φ in H^∞ let T_φ denote the operator on H^2 defined by $T_\varphi f = \varphi f$. The operator T_φ is said to be an *analytic Toeplitz operator*. In this article we consider the following questions. If an operator intertwines two analytic Toeplitz operators does it necessarily intertwine their inner parts too? Do quasisimilar analytic Toeplitz operators have equal essential spectra? Does quasisimilarity imply unitary equivalence? Although the study of Toeplitz operators has been extensive, little seems to be known about their quasisimilarity. Our purpose is to answer parts of the above questions. In particular, we give a sufficient condition for an operator intertwining two analytic Toeplitz operators to intertwine their inner parts too. We also give a sufficient condition for quasisimilarity to imply unitary equivalence.

If $\mathcal{H}$ is a separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $X : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator having trivial kernel and dense range, then X is said to be quasi-invertible. If A_1, A_2 are operators on $\mathcal{H}_1, \mathcal{H}_2$, then A_1 is *quasisimilar* to $A_2 (A_1 \sim A_2)$ if there are quasi-invertible operators $X : \mathcal{H}_1 \to \mathcal{H}_2$ and $Y : \mathcal{H}_2 \to \mathcal{H}_1$ satisfying $XA_1 = A_2X$ and $A_1Y = YA_2$. If A_1 and A_2 are unitarily equivalent we write $A_1 \cong A_2$.

Raphael [8] has shown that quasisimilar cyclic subnormal operators have equal essential spectra. Williams [9] has shown that quasisimilar quasinormal operators have equal essential spectra. However the equality of essential spectra under quasisimilarity for general subnormal operators is still open. We study the intertwining operators and suggest the pertinent questions to be considered.

A function g in H^∞ is said to be *inner* if $\lim\limits_{r\to 1} |g(re^{i\theta})| = 1$ for almost every θ. A

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function F in H^∞ is *outer* if T_F has dense range. Every φ in H^∞ has a factorization $\varphi = gF$ where g is inner and F is outer [7].

An isometry T on H^2 is a *pure isometry* if $\bigcap_{n=0}^{\infty} T^n H^2 = \{0\}$. For $g \in H^\infty$, T_g is a pure isometry if and only if g is a nonconstant inner function. For an inner function g in H^∞ we obtain the decomposition $H^2 = \sum_{k=0}^{\infty} \oplus\ g^k(H^2 \theta g H^2)$. If $\{u_k\}_{k=1}^{n}$ (n possibly ∞) is a basis for $H^2 \theta g H^2$, then $\{u_k g^m\}_{m=0}^{\infty}{}_{k=1}^{n}$ is a basis for H^2, and with respect to this basis the matrix for T_g has a block form with an (n by n) identity matrix on its subdiagonal. Also any X in the commutant $\{T_g\}'$ of T_g has a lower triangular block form which is constant along its diagonals. We denote this X by the purely formal sum $\sum_{n=0}^{\infty} \hat{X}_n T_g^n$ with X_n on the nth subdiagonal. Throughout the rest of this paper we assume φ_1 and φ_2 are in H^∞ and have the inner-outer factorization $\varphi_i = g_i F_i$ ($i = 1, 2$) unless otherwise stated.

2. **Quasisimilarity**. The following simple lemma is essential to our purposes.

LEMMA 2.1 *For $\varphi \in H^\infty$, let $\varphi = gF$ be its inner-outer factorization. Then T_φ is Fredholm if and only if T_g is Fredholm and T_F is invertible.*

PROOF. If T_g is Fredholm and T_F is invertible then it readily follows that $T_\varphi = T_g T_F$ is Fredholm. On the other hand if T_φ is Fredholm, then $\ker T_\varphi^* = \ker T_g^*$ is finite dimensional and since T_g is an isometry it follows that T_g is Fredholm. Because $T_F = T_g^* T_\varphi$, it follows that T_F is Fredholm. Hence T_F has closed range and therefore it is invertible. Q.E.D.

We would like to point out that if $\varphi = gF$ is the inner-outer factorization of φ then T_φ is Fredholm if and only if g is a finite Blaschke product and F is invertible in H^∞. The index of T_g is the negative of the number of zeros of g counting multiplicity.

Note that if $T_{\varphi_1} \sim T_{\varphi_2}$ then $T_{g_1} \cong T_{g_2}$. Indeed $T_{\varphi_1} \sim T_{\varphi_2}$ implies that $H^2 \theta g_1 H^2$ and $H^2 \theta g_2 H^2$ have the same dimension and we know that any two pure isometries of the same multiplicity are unitarily equivalent. Therefore the inner parts are strongly related to each other. In the next lemma we investigate this property further.

LEMMA 2.2. *If $X : H^2 \to H^2$ is an operator such that $XT_{\varphi_1} = T_{\varphi_2}X$ and $XT_{g_1} = T_{g_2}X$ then $XT_{F_1} = T_{F_2}X$.*

PROOF. Since $XT_{\varphi_1} = T_{\varphi_2}X$, we have $XT_{g_1}T_{F_1} = T_{g_2}T_{F_2}X$. Using the relation $XT_{g_1} = T_{g_2}X$ we get $T_{g_2}XT_{F_1} = T_{g_2}T_{F_2}X$. Because T_{g_2} is an isometry, it follows that $XT_{F_1} = T_{F_2}X$. Q.E.D.

REMARKS. Let $T_{\varphi_1} \sim T_{\varphi_2}$. If we can show that $T_{g_1} \sim T_{g_2}$ in such a way that the same quasi-invertible operators intertwining T_{φ_1} and T_{φ_2} also intertwine T_{g_1} and T_{g_2} then Lemma 2.2 shows that $T_{F_1} \sim T_{F_2}$. Using a result of Clary [1] we conclude that $\sigma(T_{F_1}) = \sigma(T_{F_2})$. Therefore T_{F_1} is invertible if and only if T_{F_2} is invertible. In other words, T_{φ_1} is Fredholm if and only if T_{φ_2} is Fredholm.

The following lemma is a slight extension of Lemma 2 of [6] and will be used in the sequel.

LEMMA 2.3. *Let* N, N' *be nilpotent operators on* K, K' *respectively and let* $X_0 = \lambda I_K + N$, $X_0' = \lambda I_{K'} + N'$ *where* λ *is a nonzero complex number. If* B, A_0, A_1, $A_2, \ldots \in \mathcal{B}(K, K')$ *satisfy*

(*a*) $\|A_k\| \leq M$, $k = 0, 1, 2, \ldots$, *and*

(*b*) $A_k X_0 = X_0' A_{k-1} + B$, $k = 1, 2, 3, \ldots$,

then $A_0 = A_1 = A_2 = \ldots$.

PROOF. Write $K = \sum_{i=1}^{n} \oplus K_i$ ($K' = \sum_{i=1}^{m} \oplus K_i'$) such that $X_0(X_0')$ has a lower triangular operator-valued matrix with diagonal entries $\lambda I_i (\lambda I_i')$ and repeat the proof of Lemma 2 of [6]. Q.E.D.

The next lemma says that the intertwining operator should be lower triangular.

LEMMA 2.4. *Assume* $XT_{\varphi_1} = T_{\varphi_2}X$ *and for* $i = 1, 2$ *write* $H_i^2 = \sum_{n=0}^{\infty} g_i^n (H^2 \theta g_i H^2)$ *then* $X : H_1^2 \to H_2^2$ *is lower triangular.*

PROOF. It suffices to show that $X^* : H_2^2 \to H_1^2$ is upper triangular. Equivalently we will show that X^* maps the subspaces $M_{2n} = \sum_{k=0}^{n} \oplus g_2^k (H^2 \theta g_2 H^2)$ into the subspaces $M_{1n} = \sum_{k=0}^{n} \oplus g_1^k (H^2 \theta g_1 H^2)$.

Now $XT_{\varphi_1} = T_{\varphi_2}X$ implies $XT_{\varphi_1}^{n+1} = T_{\varphi_2}^{n+1}X$ and hence $X^* T_{\varphi_2}^{*n+1} = T_{\varphi_2}^{*n+1} X^*$. Thus X^* maps the kernel of $T_{\varphi_2}^{*n+1}$ into the kernel of $T_{\varphi_1}^{*n+1}$. But

$$\ker T_{\varphi_i}^{*n+1} = \ker T_{g_i}^{*n+1} = H^2 \theta g_i^{n+1} H^2 = M_{in} \ (i = 1, 2)$$

Hence X is lower triangular. Q.E.D.

Note that a necessary and sufficient condition for the relation $XT_{g_1} = T_{g_2}X$ to hold is that $X : H_1^2 \to H_2^2$ be constant along its diagonals. Indeed writing T_{g_i} ($i = 1, 2$) in its block form and carrying out the necessary computations it follows that $X_{k+i,k} = X_{i0}$ $i, k = 0, 1, 2, \ldots$.

LEMMA 2.5. *Assume* $XT_{\varphi_1} = T_{\varphi_2}X$ *where* $X : H_1^2 \to H_2^2$ *is a bounded operator. If* $T_{F_1} = \sum_{n=0}^{\infty} \hat{T}_n T_{g_1}^n$ *and* $T_{F_2} = \sum_{n=0}^{\infty} \hat{T}_n' T_{g_2}^n$ *where* $T_0 = \lambda I + N$, $T_0' = \lambda I' + N'$ *with* N *and* N' *nilpotent, then* $XT_{g_1} = T_{g_2}X$.

PROOF. By Lemma 2.4 X is lower triangular. We will show that X is constant along its diagonals by inductively proving that $X_{k,0} = X_{k+1,1} = X_{k+2,2} = \ldots$ for $k = 0, 1, 2, \ldots$. We also remark that $\|X_{k+i,i}\| \leq \|X\|$ for all $k, i = 0, 1, 2, \ldots$.

If $1 \leq j < i$ then the (i, j) entry of $(XT_{g_1})T_{F_1} = T_{F_2}(T_{g_2}X)$ is

$$X_{i,j+1}T_0 + X_{i,j+2}T_1 + \ldots + X_{i,i}T_{i-j-1} = T_{i-j-1}'X_{j,j} + T_{i-j-2}'X_{j+1,j} + \ldots + T_0'X_{i-1,j}. \tag{2.6}$$

If $i = j + 1$ then $X_{j+1,j+1}T_0 = T_0'X_{j,j}$ and by Lemma 2.3 we obtain $X_{0,0} = X_{1,1} = X_{2,2} = \ldots$. To apply induction let us now assume that $X_{p,0} = X_{p+1,1} = X_{p+2,2} = \ldots$ for all $p \leq k$. Setting $i = j + k + 2$ in (2.6) we get

$$X_{k+1+j+1,j+1}T_0 = T_0'X_{k+1+j,j} + [T_1'X_{k,0} + \ldots + T_{k+1}'X_{0,0} - X_{k,0}T_1 - \ldots - X_{0,0}T_{k+1}].$$

An application of Lemma 2.3 gives us $X_{k+1,0} = X_{k+2,1} = \dots$ and hence by induction $X_{k,0} = X_{k+1,1} = \dots$ for all $k = 0, 1, 2, \dots$. Therefore $XT_{g_1} = T_{g_2}X$. Q.E.D.

The following theorem is the main result of this section. The idea of the proof is due to Deddens and Wong [6].

THEOREM 2.7. *Let* φ_1, φ_2 *be in* H^∞ *with inner-outer factorization* $\varphi_i = g_iF_i$ $(i = 1, 2)$. *Suppose there is a* λ *in* $\mathbb{C}$ *such that* g_i *factors as* $g_{i1}g_{i2}\dots g_{in}$ *and such that* $F_i - \lambda$ *is divisible by each* g_{ij}, *that is* $F_i - \lambda = g_{ij}h_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2, \dots, n$. *If* $X : H_1^2 \to H_2^2$ *is a bounded operator such that* $XT_{\varphi_1} = T_{\varphi_2}X$ *then* $XT_{g_1} = T_{g_2}X$.

PROOF. Write $T_{F_i} = \sum_{n=0}^{\infty} \hat{T}_{n,i}T_{g_i}^n$ $(i = 1, 2)$. We will show that $T_{0,i} = \lambda I_i + N_i$ where N_i is nilpotent $(i = 1, 2)$. Then Lemma 2.5 implies that $XT_{g_1} = T_{g_2}X$.

For convenience set $T_{F_i} = T_i$, $T_{g_{ij}} = T_{ij}$ and $H_i^2 = \mathcal{H}_i$. Then $T_{g_i} = T_{i1}T_{i2}\dots T_{in}$. Since $T_{0,i}^*$ is the restriction of $T_{F_i}^*$ to

$$\mathcal{H}_i\theta T_{g_i}\mathcal{H}_i = (\mathcal{H}_i\theta T_{i1}\mathcal{H}_i) \oplus T_{i1}(\mathcal{H}_i\theta T_{i2}\mathcal{H}_i) \oplus \dots \oplus T_{i1}\dots T_{in-1}(\mathcal{H}_i\theta T_{in}\mathcal{H}_i),$$

it follows that $T_{0,i}^*$ is upper triangular and hence that $T_{0,i}$ is lower triangular. Let $(T_{0,i})_{jj}$ be the compression of $T_{0,i}$ to $T_{i1}\dots T_{ij-1}(\mathcal{H}_i\theta T_{ij}\mathcal{H}_i)$. If $f, g \in \mathcal{H}_i\theta T_{ij}\mathcal{H}_i$ then since $T_i = \lambda + T_{ij}T_{h_{ij}}$ we obtain

$$\begin{aligned} T_i(T_{i1}T_{i2}\dots T_{ij-1}f) = (T_{i1}T_{i2}\dots T_{ij-1})T_if = \lambda T_{i1}T_{i2}\dots T_{ij-1}f \\ + T_{i1}T_{i2}\dots T_{ij-1}T_{ij}T_{h_{ij}}f. \end{aligned}$$

But $(T_{ij}T_{h_{ij}}T_{i1}\dots T_{ij-1}f, T_{i1}T_{i2}\dots T_{ij-1}g) = 0$, hence $(T_{0,i})_{jj} = \lambda I_j$. This shows that $T_{0,i} - \lambda I_i$ is nilpotent. Q.E.D.

COROLLARY 2.8. *Suppose* φ_1, φ_2 *are in* H^∞ *with inner-outer factorization* $\varphi_i = g_iF_i$ $(i = 1, 2)$. *If* $g_i(z) = (a - z)^n(1 - \bar{a}z)^{-n}$ $(i = 1, 2; n \geq 0, |a| < 1)$, $F_1(a) = F_2(a)$ *and* $X : H^2 \to H^2$ *is a bounded operator such that* $XT_{\varphi_1} = T_{\varphi_2}X$ *then* $XT_{g_1} = T_{g_2}X$.

PROOF. Factor $g_i = g_{i1}g_{i2}\dots g_{in}$ where $g_{ij}(z) = (a - z)(1 - \bar{a}z)^{-1}$, $i = 1, 2$ and $j = 1, 2, \dots, n$. Then $F_i - \lambda$ is divisible by each g_{ij}, where $\lambda = F_i(a)$. Applying Theorem 2.7 we obtain $XT_{g_1} = T_{g_2}X$. Q.E.D.

COROLLARY 2.9. *Suppose* $\varphi_i = z^nF_i$ $(i = 1, 2)$, $F_1(0) = F_2(0)$ *and* $XT_{\varphi_1} = T_{\varphi_2}X$. *Then* $XT_{z^n} = T_{z^n}X$.

Even though the following result looks like the uniqueness statement in the Riemann mapping theorem, we would like to point out that from the relation $T_{\varphi_1} \sim T_{\varphi_2}$ it only follows that the two sets $\varphi_1(\mathbb{D})$ and $\varphi_2(\mathbb{D})$ have the same closures and this does not convey any information about the equality of the two sets themselves. Also $G = \varphi_1(\mathbb{D})$ being simply connected might not have the property that $(\bar{G})^\circ = G$. Examples are easy to construct.

PROPOSITION 2.10. *Let* $\varphi_1 \in H^\infty$ *be univalent,* $T_{\varphi_1} \sim T_{\varphi_2}$ *and assume the normalization* $\varphi_1(0) = \varphi_2(0)$, $\varphi_1'(0) = \varphi_2'(0) > 0$ *holds. Then* $\varphi_1 = \varphi_2$.

PROOF. Since dim (ker $T^*_{\varphi_2-\lambda}$) = dim (ker $T^*_{\varphi_1-\lambda}$) = 1 or 0 for every $\lambda \in \mathbb{C}$ we conclude that the number of zeros of $\varphi_2 - \lambda$ in $\mathbb{D}$ is at most 1. Hence φ_2 is univalent. Also $XT_{\varphi_1} = T_{\varphi_2}X$ implies $XT_{\varphi_1-\varphi_1(0)} = T_{\varphi_2-\varphi_2(0)}X$. Let $\varphi_i - \varphi_i(0) = zF_i$ be the inner-outer factorizations. Since $F_1(0) = \varphi_1'(0) = \varphi_2'(0) = F_2(0)$, by Corollary 2.9 we obtain $XT_z = T_zX$. Hence $X = T_h$, h outer in H^∞. Since $XT_{\varphi_1} = T_{\varphi_2}X$ we conclude that $\varphi_1 h = \varphi_2 h$, so $\varphi_1 = \varphi_2$.

3. **Unitary Equivalence**. In this section we consider the following problem. If two analytic Toeplitz operators are quasisimilar, must they be unitarily equivalent? The answer to this question is still unknown. However, in certain special cases an improvement is possible. For example, Conway [3] has shown that if S is the unilateral shift of multiplicity one, $S \sim T_\varphi$ then $S \cong T_\varphi$ ($\varphi \in H^\infty$). We use results of Conway ([2], [4, p. 220] and Clary [1]) to obtain an extension of a result of Cowen [5]. Note that if u is an inner function, we say that the *order of* u is n if u is a finite Blaschke product of order n; otherwise we say the order of u is infinity. For any operator $A \in \mathcal{B}(\mathcal{H})$, let $P^\infty(A)$ denote the weak * closed subalgebra of $\mathcal{B}(\mathcal{H})$ generated by A. That is, $P^\infty(A)$ is the weak * closure in $\mathcal{B}(\mathcal{H})$ of the polynomials in A.

THEOREM 3.1. *Suppose* φ *and* ψ *are in* H^∞ *and there are inner functions* u *and* v *such that* $P^\infty(T_\varphi) = P^\infty(T_u)$ *and* $P^\infty(T_\psi) = P^\infty(T_v)$. *Then the following are equivalent:*

(*a*) $T_\varphi \cong T_\psi$,

(*b*) $T_\varphi \sim T_\psi$,

(*c*) *there are functions* g *in* H^∞ *and* w_1, w_2 *inner such that* $\varphi = g \circ w_1$, $\psi = g \circ w_2$ *and order* w_1 = *order* w_2.

PROOF. (*c*) implies (*a*) follows from Theorem 1 of [5], and (*a*) implies (*b*) is clearly true, so we only need to prove that (*b*) implies (*c*).

By hypothesis, there are inner functions u and v so that $P^\infty(T_\varphi) = P^\infty(T_u)$ and $P^\infty(T_\psi) = P^\infty(T_v)$. We will show that u and v have the same order. Let n = order u and m = order v. It is easy to see that $P^\infty(T_\varphi) = P^\infty(T_u) = \{T_{g\circ u} : g \in H^\infty\}$ and $P^\infty(T_\psi) = P^\infty(T_v) = \{T_{g\circ v} : g \in H^\infty\}$. Let X and Y be quasi-invertible operators such that $XT_\varphi = T_\psi X$ and $YT_\psi = T_\varphi Y$. By a result of Conway [2] (see also [4, p. 220]) there is an isometric isomorphism $F : P^\infty(T_\varphi) \to P^\infty(T_\psi)$ having the following properties

(1) $F(T_\varphi) = T_\psi$,

(2) $XA = F(A)X$ and $YF(A) = AY$ for every A in $P^\infty(T_\varphi)$, and

(3) F is a weak * homeomorphism.

Now F induces an algebra isomorphism $\Phi : H^\infty \circ u \to H^\infty \circ v$ given by $\Phi(g \circ u) = h \circ v$ where $T_{h\circ v} = F(T_{g\circ u})$. Let $w = \Phi(u) = q \circ v$ and $w_0 = \Phi^{-1}(v) = q_0 \circ u$. Since Φ is weak * continuous we have $v = \Phi(w_0) = \Phi(q_0 \circ u) = q_0 \circ w = q_0 \circ q \circ v$. Because v is inner, we have $q_0(q(z)) = z$, z in $\mathbb{D}$. Also $w = q \circ v = q \circ \Phi(w_0) = q \circ \Phi(q_0 \circ u) = q \circ q_0 \circ w$. Moreover since $T_u \sim T_w$, we have by a result of Clary [1], $\sigma(T_u) = \sigma(T_w)$ Hence $\overline{w(\mathbb{D})} = \overline{u(\mathbb{D})} = \overline{\mathbb{D}}$. Therefore $q(q_0(z)) = z$, z in $\mathbb{D}$. But

$\overline{q(\mathbb{D})} = \overline{q(v(\mathbb{D}))} = \overline{w(\mathbb{D})} = \overline{\mathbb{D}}$ and $\overline{q_0(\mathbb{D})} = \overline{\mathbb{D}}$. Hence q is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ and w is an inner function of order m. But $T_u \sim T_v$ implies dim (ker T_u^*) = dim (ker T_w^*). Thus order u = order w, so $n = m$.

Since $T_\varphi \in P^\infty(T_\varphi)$, there is $g \in H^\infty$ such that $\varphi = g \circ u$. Since Φ is weak * continuous we have $\Phi(\varphi) = g \circ \Phi(u) = g \circ w$, but we have $F(T_\varphi) = T_\psi$ so $\Phi(\varphi) = \psi$. Therefore, we have $\varphi = g \circ u$ and $\psi = g \circ w$ where $g \in H^\infty$ and order u = order $w = n$ so the conclusion follows with $w_1 = u$ and $w_2 = w$.

Q.E.D.

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THE DIRICHLET PROBLEM WITH DENJOY-PERRON INTEGRABLE BOUNDARY CONDITION

BY

M. BENEDICKS AND W. F. PFEFFER

Dedicated to the memory of Professor J. B. Diaz

ABSTRACT. The Poisson integral of a Denjoy-Perron integrable function defined on the boundary of an open disc is harmonic in this disc. Moreover, almost everywhere on the boundary, the nontangential limits of the integral coincide with the boundary condition. This extends the classical result for Lebesgue integrable boundary conditions. By means of conformal maps, a generalization to domains bounded by a sufficiently smooth Jordan curve is also obtained.

By R and C we shall denote the sets of all real and complex numbers, respectively. We let $D = \{z \in C: |z| < 1\}$ and $T = \{z \in C: |z| = 1\}$.

It is well known that a convolution of the Poisson kernel with a Lebesgue integrable function f on T produces a harmonic function in D whose nontangential limits are equal to f almost everywhere on T (see, e.g., [11], thm. IV.1). By a limiting procedure this result can be extended to the case when f has only an improper Lebesgue integral. While the idea of such an extension is straightforward, the actual proof, which involves a switch of iterated limits, is unpleasant. How to proceed when the singularities of f cluster so that the improper Lebesgue integral does not exist (see later Example) is unclear.

Since the Denjoy-Perron integral (abbreviated as DP integral) generalizes the improper Lebesgue integral and is, in fact, closed with respect to the formation of improper integrals, it seems appropriate to consider functions which are DP integrable on T. We are obliged to J. B. Diaz for suggesting this to the second author. The second author is also obliged to A. K. Lyzzaik for several stimulating discussions.

To convince the reader that using the DP integral is not extravagant, we briefly recall the Kurzweil and Henstock definition (see [7] and [6]), which shows that the DP integral is only a very natural generalization of the classical Riemann integral.

A *partition* of an interval $[a, b] \subset R$ is a set

$$\{t_0, \ldots, t_n; \tau_1, \ldots, \tau_n\}$$

such that $a = t_0 < \ldots < t_n = b$, and $t_{j-1} \leq \tau_j \leq t_j$ for $j = 1, \ldots, n$. If δ is a strictly

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positive function on $[a, b]$, we say that the partition $\{t_0, \ldots, t_n; \tau_1, \ldots, \tau_n\}$ of $[a, b]$ is δ-*fine* whenever $t_j - t_{j-1} < \delta(\tau_j)$ for $j = 1, \ldots, n$. A function $f: [a, b] \rightarrow R$ is called *DP integrable* if there is a real number $\int_b^a f(t)\,dt$ such that given $\epsilon > 0$, we can find a positive function δ on $[a, b]$ with

$$\left| \sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_b^a f(t)\,dt \right| < \epsilon$$

for each δ-fine partition $\{t_0, \ldots, t_n; \tau_1, \ldots, \tau_n\}$ of $[a, b]$.

Basic properties of the DP integral are derived from the above definition in [8]. They include the relationship between the DP and Lebesgue integrals, and also the equivalence of the Kurzweil-Henstock and Perron definitions of the DP integral.

Since DP integrable functions are in general not absolutely integrable, the usual *maximal function* approach to the Dirichlet problem (see [5], Chpt. 2, Sec. 3) cannot be applied. However, following the proof of ([11], Thm. IV.1), which essentially goes back to Fatou, we show that a judicious use of integration by parts for the DP integral (see [9] for a simple proof) yields a result completely analogous to the classical one. The critical property we employ is that almost everywhere the derivative of an indefinite DP integral exists and equals the integrand (see [10], Chpt. VI, Thm. (6.1)).

The usefulness of our result is illustrated by an example of a DP integrable function which appears unmanageable within the limits of Lebesgue integration.

Throughout this paper, integrability always means DP integrability, and all integrals are DP integrals. A function $f: T \rightarrow R$ is called integrable if the integral $\int_{-\pi}^{\pi} f(e^{i\theta})\,d\theta$ exists.

The *Poisson kernel* is the positive function

$$P_r(t) = \frac{1}{2\pi} \cdot \frac{1 - r^2}{1 - 2r\cos t + r^2}$$

defined for $0 \leq r < 1$ and $t \in R$. Instead of $(\partial/\partial t)P_r(t)$ we shall write $P_r'(t)$. We note that by associating $P(z) = P_r(t)$ to each $z = re^{it}$ in D, we can define a map $P: D \rightarrow R$.

PROPOSITION 1. *Let $f: T \rightarrow R$ be an integrable function. Then the integral*

$$u(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\,dt$$

exists for each $(r, \theta) \in [0, 1) \times R$, and the function u is harmonic in D.

PROOF. For $t \in R$, let $F(t) = \int_{-\pi}^{t} f(e^{i\theta})\,d\theta$. Integrating by parts (see [9]), we see that the Poisson integral defining the function u exists, and that

$$u(re^{i\theta}) = P_r(\theta - \pi)F(\pi) + \int_{-\pi}^{\pi} P_r'(\theta - t)F(t)\,dt.$$

Now F is a continuous function, and P_r and P_r' are harmonic in D. Hence the proposition follows by differentiation under the integral sign (note that this is the Lebesgue integral).

For the remainder of this note, we shall assume that $f: T \to R$ is an integrable function. For $(r, t) \in [0, 1) \times R$ we let

$$F(t) = \int_0^t f(e^{i\theta})\, d\theta$$

$$P * f(r, e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\, dt.$$

PROPOSITION 2. *Let $\theta_0 \in [-\pi, \pi]$ be such that $F'(\theta_0) = f(e^{i\theta_0})$. If $u = P * f$, then*

$$u(re^{i\theta}) \to f(e^{i\theta_0})$$

whenever $r \to 1-$, and there is a $c > 0$ such that $|\theta - \theta_0| \leq c(1 - r)$.

PROOF. It suffices to prove the special case of $f(e^{i\theta_0}) = \theta_0 = 0$; for this yields the general result when applied to the function

$$g(z) = f(ze^{i\theta_0}) - f(e^{i\theta_0}).$$

By the continuity of F,

$$M = \sup\{|F(t)| : |t| \leq \pi\} < +\infty.$$

Choose $c > 0$, $\epsilon > 0$, and let

$$\eta = \epsilon(5M + c + 1)^{-1}.$$

As $F'(0) = 0$, there is a $\delta \in (0, \pi/2)$ such that $|F(t)| \leq \eta|t|$ for each $t \in [-\delta, \delta]$. Find a $\rho \in (0, 1)$ so that $c(1 - \rho) \leq \delta/2$ and $P_r(t) \leq \eta$ whenever $\rho \leq r < 1$ and $\delta/2 \leq |t| \leq \pi$. We show that if $\rho \leq r < 1$ and $|\theta| \leq c(1 - r)$, then $|u(re^{i\theta})| \leq \epsilon$.

Let $\rho \leq r < 1$ and $|\theta| \leq c(1 - r) \leq \delta/2$. An integration by parts yields

$$|u(re^{i\theta})| = \left| P_r(\theta - \pi)F(\pi) - P_r(\theta + \pi)F(-\pi) + \int_{-\pi}^{\pi} P_r'(\theta - t)F(t)\, dt \right|$$

$$\leq 2M\eta + M\int_{\delta \leq |t| \leq \pi} |P_r'(\theta - t)|\, dt + \eta \int_{-\delta}^{\delta} |P_r'(\theta - t)t|\, dt.$$

Observing that $P_r'(t)\sin t \leq 0$, we shall estimate each integral separately. Since $P_r'(-t) = -P_r'(t)$, both integrals are even functions of θ, and hence it suffices to estimate them for $\theta \geq 0$. Given $\theta \geq 0$, we have

$$\int_{\delta \leq |t| \leq \pi} |P_r'(\theta - t)|\, dt = \int_{\delta}^{\pi} P_r'(\theta - t)\, dt + \int_{-\pi}^{-\pi + \theta} P_r'(\theta - t)\, dt$$

$$- \int_{-\pi + \theta}^{-\delta} P_r'(\theta - t)\, dt = P_r(\theta - \pi) + 2P_r(\pi)$$

$$- P_r(\theta - \delta) - P_r(\theta + \pi) - P_r(\theta + \delta) \leq 3\eta,$$

and similarly

$$\int_{-\delta}^{\delta} |P_r'(\theta - t)t \, dt| = \int_{-\delta}^{0} P_r'(\theta - t)t \, dt - \int_{0}^{\theta} P_r'(\theta - t)t \, dt$$

$$+ \int_{\theta}^{\delta} P_r'(\theta - t)t \, dt = 2\theta P_r(0) - \delta P_r(\theta + \delta) - \delta P_r(\theta - \delta)$$

$$+ \int_{-\delta}^{0} P_r(\theta - t) \, dt - \int_{0}^{\theta} P_r(\theta - t) \, dt + \int_{\theta}^{\delta} P_r(\theta - t) \, dt$$

$$\leq 2\theta P_r(0) + \int_{-\pi}^{\pi} P_r(\theta - t) \, dt = \frac{\theta}{\pi} \frac{1 + r}{1 - r} + 1 \leq c + 1;$$

for $1 + r < 2$, and $\theta < c(1 - r)$. Combining the previous estimates, we see that

$$|u(re^{i\theta})| \leq \eta(5M + c + 1) = \epsilon,$$

and the proposition is proved.

The convergence $u(re^{i\theta}) \to f(e^{i\theta_0})$ described in Proposition 2 is usually referred to as *nontangential convergence*. Using ([10], Chpt. VI, Thm. (6.1)), we can summarize our results.

THEOREM 1. *If f is a DP integrable function on T, then $P * f$ is harmonic in D, and it converges nontangentially to f almost everywhere in T.*

EXAMPLE. Let $I = (a, b)$ with $a, b \in R$ and $a < b$. Set

$$F_I(t) = (t - a)^2 \cdot (t - b)^2 \cdot \sin \frac{\pi}{2} \left(\frac{b - a}{t - a}\right)^2 \cdot \sin \frac{\pi}{2} \left(\frac{b - a}{t - b}\right)^2$$

if $t \in I$, and $F_I(t) = 0$ if $t \in R - I$. Then the finite derivative $F_I'(t)$ exists for each $t \in R$, and it is easy to see that F_I' is not Lebesgue integrable over I (though, the improper Lebesgue integral exists).

Let K be a Cantor discontinuum in the interval $[-\pi, \pi]$, and let I_n, $n = 1, 2, \ldots,$ be the connected components of $(-\pi, \pi) - K$. As

$$|F_I(t)| \leq (t - a)^2(t - b)^2,$$

it is not difficult to check that the function $F = \sum_{n=1}^{\infty} F_{I_n}$ has a finite derivative F'. Now define $f: T \to R$ by setting $f(z) = F'(\arg z)$ for each $z \in T$. Then f is DP integrable, and $F(t) = \int_{-\pi}^{t} f(e^{i\theta}) \, d\theta$ for each $t \in R$ (see [8], Thm. A_3). Consequently, $P * f$ is a harmonic function in D whose nontangential limits are equal to f everywhere in T.

However, $f(e^{it})$ is not Lebesgue integrable in any open interval containing points of K. In particular, $f(e^{it})$ does not have an improper Lebesgue integral over $[-\pi, \pi]$.

REMARK 1. Following our proofs, it is easy to verify that Theorem 1 holds for each function $f: T \to R$ for which we can find a bounded measurable function F with $F'(t) = f(e^{it})$ for almost all $t \in [-\pi, \pi]$. Indeed, we only need to define $P * f$ in the distribution sense, i.e., by setting

$$P * f(re^{i\theta}) = P_r(\theta - \pi)F(\pi) + \int_{-\pi}^{\pi} P_r'(\theta - t)F(t) \, dt$$

for each $re^{i\theta} \in D$. This, of course, links with the standard fact of harmonic analysis that the convolution of P and a tempered distribution is harmonic in D. However, it is the important feature of DP integrable functions that the convolution $P * f$ can be defined directly.

Using conformal maps, Theorem 1 can be extended to more general domains. We shall describe this next.

As usual, we say that a map $\gamma : T \to C$ is of class $\mathscr{C}^{1,\epsilon}$, $0 < \epsilon \le 1$, if it has a derivative which is Lipschitz of order ϵ (see [3], Sec. 5.1).

Let Ω be a *Jordan domain* of class $\mathscr{C}^{1,1}$, i.e., a bounded open subset of $\mathscr{C}$ whose boundary $\partial\Omega$ is a Jordan curve of class $\mathscr{C}^{1,1}$, and let $\gamma : [0, a] \to \partial\Omega$ be a counter-clockwise parametrization of $\partial\Omega$ by arclength. A function $f : \partial\Omega \to R$ is called integrable if the integral $\int_0^a f[\gamma(s)]\,\mathrm{d}s$ exists. By ([8], Coroll. A6), the integrability of f does not depend on the parametrization γ of $\partial\Omega$.

By the Riemann mapping theorem and the Caratheodory extension theorem (see [11], Thm. IX.2), there is a homeomorphism $\phi : D \cup T \to \Omega \cup \partial\Omega$ such that $\phi(T) = \partial\Omega$ and ϕ is conformal in D. Each such homeomorphism is called a *Riemann map*.

THEOREM 2. *Let Ω be a Jordan domain of class $\mathscr{C}^{1,1}$, let f be a DP integrable function on $\partial\Omega$, and let $\phi : D \cup T \to \Omega \cup \partial\Omega$ be a Riemann map. Then $f \circ \phi$ is a DP integrable function on T. Moreover, $[P * (f \circ \phi)] \circ \phi^{-1}$ is harmonic in Ω, and it converges nontangentially to f almost everywhere in $\partial\Omega$.*

PROOF. Let $\gamma : [0, a] \to \partial\Omega$ be a counter-clockwise parametrization of $\partial\Omega$ by arclength such that $\gamma(0) = \gamma(a) = \phi(-1)$. By Kellog's theorem (see [11], Thm. IX.7), the derivative ϕ' can be extended continuously from D to $D \cup T$, and this extension (also denoted by ϕ') is nowhere equal to zero. Letting

$$s(\theta) = \int_{-\pi}^{\theta} |\phi'(e^{it})|\,\mathrm{d}t,$$

we have $\gamma[s(t)] = \phi(e^{it})$ for each $t \in [-\pi, \pi]$. The change of variable theorem (see [8], Coroll. A6) yields

$$\begin{aligned}\int_0^a f[\gamma(s)]\,\mathrm{d}s &= \int_{-\pi}^{\pi} f \circ \gamma[s(t)]s'(t)\,\mathrm{d}t \\ &= \int_{-\pi}^{\pi} f \circ \phi(e^{it})|\phi'(e^{it})|\,\mathrm{d}t,\end{aligned}$$

and so $f \circ \phi \cdot |\phi'|$ is integrable. The integrability of $f \circ \phi$ is now a consequence of the following claim.

CLAIM. *The function $\phi'(e^{it})$ is absolutely continuous in $[-\pi, \pi]$.*

Indeed, as $|\phi'| > 0$ is continuous in $D \cup T$, the claim implies that $|\phi'(e^{it})|^{-1}$ has a finite variation in $[-\pi, \pi]$. By the Proposition in [9], this guarantees the integrability of

$$f \circ \phi = [f \circ \phi \cdot |\phi'|] \cdot |\phi'|^{-1}.$$

Now giving the obvious meaning to "nontangential convergence" and "almost everywhere" in $\partial\Omega$, and using the conformity of ϕ, Theorem 2 follows from Theorem 1.

It remains to establish the claim.

Proof of the Claim. For $|z| \leq 1$, let

$$G(z) = \log \phi'(z) = u(z) + iv(z)$$

where $u(z) = \log |\phi'(z)|$, and $v(z) = \arg \phi'(z) = -i \log (\phi'(z))/(|\phi'(z)|)$. If $g(t) = v(e^{it})$, $t \in [-\pi, \pi]$, then by [3], Sec. 1.1,

$$v(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)g(t)\,dt$$

for each $z = re^{i\theta}$ in D. By differentiating the equation

$$\gamma[s(t)] = \phi(e^{it}),$$

$t \in [-\pi, \pi]$, we obtain

$$\gamma'[s(t)] \cdot |\phi'(e^{it})| = ie^{it}\phi'(e^{it}),$$

and hence

$$g(t) = -i \log \{-ie^{it}\gamma'[s(t)]\}.$$

Since γ is of class $\mathscr{C}^{1,1}$ and $s'(t) = |\phi'(e^{it})|$ is bounded, we see that g is Lipschitz of order 1. In particular, g has a bounded derivative g' almost everywhere in $[-\pi, \pi]$. For $z = re^{i\theta}$ in D, we have

$$\begin{aligned} iz\frac{\phi''(z)}{\phi'(z)} = izG'(z) &= \frac{\partial}{\partial\theta}G(z) \\ &= \frac{\partial}{\partial\theta}u(z) + i\int_{-\pi}^{\pi} P_r'(\theta - t)g(t)\,dt \\ &= \frac{\partial}{\partial\theta}u(z) + i\int_{-\pi}^{\pi} P_r(\theta - t)g'(t)\,dt. \end{aligned}$$

As g' is essentially bounded,

$$w(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)g'(t)\,dt$$

is bounded in D, and hence w is of Hardy class $h^\infty \subset h^2$ (for the definition of Hardy's classes, we refer to [3], Sec. 1.1). An application of M. Riesz' theorem (see [3], Thm. 4.1) shows that G', and hence $\phi'' = \phi' G'$ is in Hardy class $H^2 \subset H^1$. Now the claim follows from [3], Thm. 3.11.

REMARK 2. If the function f is Lebesgue integrable on $\partial\Omega$, then it is easy to see that Theorem 2 holds whenever Ω is a Jordan domain of class $\mathscr{C}^{1,\epsilon}$. Recently, it was shown in [1], [2], and by a different method in [4], that $\mathscr{C}^{1,\epsilon}$ can be relaxed to $\mathscr{C}^1$ (i.e., continuously differentiable boundary with no Lipschitz conditions imposed) if $|f|^p$ is Lebesgue integrable on $\partial\Omega$ for some $p > 1$. If f is merely DP integrable, this generality is not possible for the following reason: a function which is Lipschitz of order $\epsilon < 1$ may be of infinite variation, and hence its product with a DP integrable function need not be DP integrable.

REMARK 3. If $\partial\Omega$ is only *piecewise* $\mathscr{C}^{1,1}$ it is not difficult to show that Theorem 2 still holds provided Ω is *locally convex* at each corner of $\partial\Omega$. The same condition must be imposed even if the function f is Lebesgue integrable on $\partial\Omega$.

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STRUCTURE OF CERTAIN PERIODIC RINGS

BY

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ABSTRACT. Let R be a periodic ring, N the set of nilpotents, and D the set of right zero divisors of R. Suppose that (i) N is commutative, and (ii) every x in R can be *uniquely* written in the form $x = e + a$, where $e^2 = e$ and $a \in N$. Then N is an ideal in R and R/N is a Boolean ring. If (i) is satisfied but (ii) is now assumed to hold merely for those elements $x \in D$, and if $1 \in R$, then N is still an ideal in R and R/N is a subdirect sum of fields. It is further shown that if (i) is satisfied but (ii) is replaced by: "every right zero divisor is either nilpotent or idempotent," and if $1 \in R$, then N is still an ideal in R and R/N is either a Boolean ring or a field.

Throughout, N denotes the set of nilpotents and D denotes the set of right zero divisors of R. The ring R is called *periodic* if for every x in R, there exist distinct positive integers $m = m(x)$, $n = n(x)$ such that $x^m = x^n$. A Boolean ring is trivially a periodic ring with commuting nilpotents and, of course, every x in R can be *uniquely* written as a sum of an idempotent and a nilpotent. That these properties are *not* confined just to Boolean rings can be seen by considering the ring of integers, modulo 4. In Theorem 1 below, we show that a periodic ring R with the above properties, while not necessarily Boolean, is the next best thing to being Boolean in the sense that its factor ring R/N is indeed Boolean (and hence a subdirect sum of copies of GF(2)). Next, we consider a periodic ring R with identity 1 and with commuting nilpotents such that every right zero divisor x can be *uniquely* written in the form $x = e + a$, where $e^2 = e$ and $a \in N$. Here again N turns out to be an ideal in R but R/N is now a subdirect sum of (not necessarily identical) fields. On the other hand, if we replace the last hypothesis above by "every right zero divisor is either nilpotent or idempotent," then N is still an ideal in R and R/N is now necessarily a Boolean ring or a field.

We begin this note with the following

THEOREM 1. *Let R be a periodic ring (not necessarily with identity). Suppose that* (i) *N is commutative, and* (ii) *every x in R can be uniquely written in the form* $x = e_0 + a_0$, *where* $e_0^2 = e_0$ *and* $a_0 \in N$. *Then N is an ideal in R, and R/N is Boolean (and hence a subdirect sum of copies of* GF(2)). *In fact, R is commutative.*

PROOF. Let $e^2 = e \in R$, $x \in R$, and let $f = e + ex - exe$. Then $f^2 = f$. Moreover, since

$$f = e + (ex - exe); \quad ex - exe \in N;$$

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and

$$f = f + 0,$$

it follows from (ii) that $ex - exe = 0$, and hence $ex = exe$. Similarly, $xe = exe$, and thus

(1) All idempotents of R are central.

Combining (1) with hypotheses (ii) and (i), we see that R is commutative and hence N is an ideal in R. Let $x \in R$. By (ii),

$$x = e_0 + a_0; \qquad e_0^2 = e_0, \qquad a_0 \in N,$$

and hence $x + N$ is idempotent. Thus, R/N is Boolean.

THEOREM 2. *Let R be a periodic ring with identity 1. Suppose that (i) N is commutative, (ii) every $x \in D$ can be uniquely written in the form $x = e + a$, where $e^2 = e$ and $a \in N$. Then N is an ideal in R and R/N is isomorphic to a subdirect sum of fields.*

PROOF. Let $e^2 = e \in R$, $x \in R$, and let $f = e + ex - exe$. If $f = 1$, then $ex = exe$. Now, suppose $f \neq 1$. Then, $f^2 = f$, $f \neq 1$, and hence $f \in D$. Since

$$f = e + a, \quad \text{where } a = ex - exe \in N;$$

and

$$f = f + 0,$$

it follows from (ii) that $a = 0$ (since $f \in D$), and thus $ex = exe$. Similarly, $xe = exe$, and hence

(2) All idempotents are central.

Let $x \in R$. Since R is periodic, $x^m = x^n$ for some integers $m > n \geqq 1$, and hence $x^{(m-n)n}$ is idempotent. Therefore, by (2), for all y in R,

$$[x^{(m-n)n}, y] = 0 \tag{3}$$

where $[u, v] = uv - vu$. A well known Theorem of Herstein [2] asserts that (3) implies that the commutator ideal of R is nil and hence the nilpotents N of R form an ideal in R. Also, since $x^m = x^n$, for some polynomial $g(\lambda) \in \mathbb{Z}[\lambda]$,

$$(x^{m-n+1} - x)^n = (x^{m-n+1} - x)x^{n-1}g(x) = 0$$

and hence $x^{m-n+1} - x \in N$, $m > n \geqq 1$. Thus,

$$(x + N)^{m-n+1} = x + N;\ m - n + 1 > 1, \qquad x \in R. \tag{4}$$

By a well known theorem of Jacobson [3], (4) implies that R/N is a subdirect sum of fields.

THEOREM 3. *Let R be a periodic ring with identity 1. Suppose that (i) N is commutative, and (ii) every x in D is either idempotent or nilpotent. Then N is an ideal of R, and R/N is either Boolean or a field.*

PROOF. Suppose $x \in R$, $x \notin D$. Since R is periodic, let $x^m = x^n$, $m > n \geqq 1$. Then $(x^{m-n} - 1)x^n = 0$. Since $x \notin D$, $x^{m-n} - 1 = 0$, and hence by (ii),

(5) For every x in R, x is nilpotent or idempotent or a unit.

CLAIM A. *If $a \in N$ and e is an idempotent, then $ae \in N$ and $ea \in N$.*

PROOF. Since N is commutative, we have

(6) N is a subring of R.

Let $a \in N$ and $e^2 = e$. Then $ae - eae \in N$, and hence by (i) we have $(ae - eae)a = a(ae - eae)$. So

$$aea - eaea = a^2e - aeae. \tag{7}$$

Multiplying (7) by e from left and right we get $ea^2e = eaeae$. So $(eae)^2 = eaeeae = ea^2e$. Hence we have shown that

$$(eae)^2 = ea^2e \text{ for every } a \in N, \tag{8}$$

and every idempotent e in R.

If $(eae)^{2^k} = ea^{2^k}e$, then $(eae)^{2^{k+1}} = (ea^{2^k}e)^2 = ea^{2^{k+1}}e$ (by (8)). The above induction shows that

$$(eae)^{2^n} = ea^{2^n}e \tag{9}$$

for all positive integers n.

Since $a \in N$, (9) implies that $eae \in N$. But $ae - eae \in N$, and hence, by (6), we get $ae \in N$. Similarly, $ea \in N$. This proves Claim A.

CLAIM B. *Let $a \in N$ and x be a unit in R. Then $ax \in N$ and $xa \in N$.*

PROOF. Suppose $ax \notin N$. Then

$$ax \neq xa. \tag{10}$$

Also, ax is not a unit in R (since a is nilpotent and x is invertible). So ax is idempotent, by (5), and hence $axax = ax$. So

$$axa = a \text{ (since } x \text{ is invertible).} \tag{11}$$

Now, $1 + x \notin N$, since $a(1 + x) \neq (x + 1)a$ and N is commutative. If $(1 + x)^2 = 1 + x$, then $x^2 = -x$. So $x = -1$, which contradicts (10). Hence $(1 + x)$ is not idempotent, and since $1 + x \notin N$, we get from (5) that

(12) $1 + x$ is a unit in R.

Since $ax \notin N$, it follows that $a(1 + x) = a + ax \notin N$. Clearly, by (12), $a(1 + x)$ is not a unit in R, and hence $a(1 + x)$ is idempotent, by (5). Thus, $(a + ax)^2 = a + ax$. So $a^2 + a^2x + axa + (ax)^2 = a + ax$. Using (11) and $(ax)^2 = ax$ we get $a^2(1 + x) = 0$. Then (12) implies that

$$a^2 = 0. \tag{13}$$

Since $a \in N$ and $x^{-1}ax \in N$, therefore, by (i) and (11), $a(x^{-1}ax) = (x^{-1}ax)a = x^{-1}(axa) = x^{-1}a$. Hence

$$ax^{-1}ax = x^{-1}a. \tag{14}$$

Multiplying (14) by a from the left, and using (13) we get $ax^{-1}a = 0$. Then (14) implies that $x^{-1}a = 0$. Hence $a = 0$, which contradicts (10). Therefore, $ax \in N$. Similarly, $xa \in N$ and Claim B is proved.

Now we can complete the proof of Theorem 3. Clearly, since N is commutative, the product of two nilpotent elements is nilpotent. So it follows from (5) and Claims A and B that N is an ideal of R.

Let $x + N$ be any nonzero right zero divisor in R/N. Then $(y + N)\cdot(x + N) = N$, $x \notin N$, $y \notin N$. Thus $yx + N = N$, and hence

$$yx \in N, \qquad x \notin N, \qquad y \notin N. \tag{15}$$

Note that x is not a unit; otherwise, $y \in N$ (see (15)). Thus, by (5) x is idempotent and hence $(x + N)^2 = x^2 + N = x + N$. This shows that

(16) Every right zero divisor of R/N is idempotent.

Moreover, by (5), we see that

(17) Every $x + N$ in R/N is idempotent or a unit in R/N.

Claim C. *If R/N has an idempotent different from N and $1 + N$, then R/N is Boolean.*

Proof. Let $(f + N)^2 = f + N; f \notin N; f - 1 \notin N$. Suppose $u + N$ is not idempotent. Then, by (17), $u + N$ is a unit in R/N and, of course, $u + N \neq 1 + N$. Note that $(f + N)(u + N)$ is not a unit in R/N; otherwise, $f + N$ would be a unit in R/N. Hence, by (17),

$$(f + N)(u + N) \text{ is idempotent.} \tag{18}$$

Now, since R/N is periodic and has no nonzero nilpotents, by a well known theorem of Herstein [1], R/N is commutative. Combining this with (18), we see that

$$(f + N)(u + N) = \{(f + N)(u + N)\}^2 = fu^2 + N$$

and hence $f(u - u^2) + N = N$. But $u + N$ is a unit and hence $f(1 - u) + N = N$. Thus, $(1 - u) + N$ is a right zero divisor (since $f \notin N$), and hence by (16), $(1 - u) + N$ is idempotent. Thus, $u^2 + N = u + N$ and hence $u + N = 1 + N$, a contradiction. This contradiction proves Claim C. Combining (17), Claim C, and the fact that R/N is commutative, the theorem follows.

In conclusion, we would like to express our indebtedness and gratitude to the referee for his helpful comments and valuable suggestions.

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THE FIBRE OF THE DOUBLE SUSPENSION IS AN H-SPACE

BY

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ABSTRACT. In this paper we show that the homotopy-theoretic fibre of the double suspension map $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$ is an H-space.

1. **Introduction**. Let $C(n)$ denote the homotopy-theoretic fibre of the double suspension map $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$. The goal of this paper is the proof of the following theorem.

THEOREM 1.1. *$C(n)$ is an H-space for all n.*

This lends support to Mahowald's conjecture that $C(n)$ is a loop space for all n and perhaps even double loop space. If $n = 1$, the theorem is trivial since $C(1) = \Omega_0^3 S^3$, so from now on we will assume $n > 1$. The only other integers for which S^{2n-1} is an H-space are $n = 2$ and $n = 4$, but even in these cases E^2 is not an H-map. However, if we invert the prime 2, then S^{2n-1} becomes an H-space and E^2 becomes an H-map for all n so the theorem has content only at the prime 2. Thus from now on we will assume all spaces and maps to be localized at 2.

The proof is based on Sugawara's criterion [9] that a space X is an H-space if and only if the canonical map $\alpha: X \to \Omega SX$ has a retraction up to homotopy. We construct a map $r: \Omega SC(n) \to C(n)$ and prove that r induces an isomorphism on homotopy in degree $4n - 3$ (the least non-vanishing degree). We then conclude the proof by appealing to the following theorem which is proved in Section 3.

THEOREM 3.1. *Let $f: C(n) \to C(n)$ be such that f induces an isomorphism on $H_{4n-3}(C(n); Z/2Z)$, $n > 1$. Then f is a homotopy equivalence. (Recall that we have localized at 2.)*

The idea that it is useful in homotopy theory to show that certain spaces have this property (that any self-map which induces an isomorphism on the lowest non-vanishing homotopy group must be a homotopy equivalence) is due to Fred Cohen. It was first introduced into the literature by Cohen, Moore and Neisendorfer in [6] where the property was given the name "atomic". The fact that $C(n)$ is atomic for $n > 1$ is certainly well known to all the authors of [3]. The proof given in Section 3 is a straightforward application of the ideas used by Cohen and Mahowald in [5].

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REMARK. Fred Cohen has shown that $C(1)$ is also atomic at the prime 2 but the proof is more difficult in this case.

The fact that $C(2)$ is an H-space also follows from the decomposition

$$\Omega^2 S^5\{2\} \simeq \Omega^2 S^3\langle 3\rangle \times C(2)$$

recently proved by Fred Cohen, where $X\{2\}$ denotes the homotopy-theoretic fibre of the squaring map on the H-space X and $S^3\langle 3\rangle$ denotes the 3-connective cover of S^3. In fact, Fred Cohen and the author have recently shown that

$$\Omega^3 S^9\{2\} \simeq C(2) \times \Omega C(4)$$

which shows that $C(2)$ is a homotopy abelian H-space. However, it is shown in [3] that a decomposition of this form will not be possible for arbitrary n.

2. **Proof of the main theorem**. In this section, we give the proof of the main theorem modulo some atomicity statements contained in Section 3. Throughout this section, $n > 1$ and all spaces and maps have been localized at 2.

Let j denote the homotopy class of the inclusion of the fibre into the total space of the fibration $C(n) \xrightarrow{j} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$. The double adjoint of $E^2 \circ j$ is $S^2 j : S^2 C(n) \to S^{2n+1}$ so $S^2 j \simeq 0$. Let $H : \Omega S X \to \Omega S(X \wedge X)$ denote the second Hopf-invariant map. By naturality of the Hopf-invariant map, we have a homotopy commutative diagram

$$\begin{tikzcd}
 & \Omega S C(n) \arrow[r, "H"] \arrow[d, "\Omega S j"] & \Omega S(C(n) \wedge C(n)) \arrow[d, "\Omega S(j \wedge j)"] \\
S^{2n-1} \arrow[r, "E"] & \Omega S^{2n} \arrow[r, "H"] & \Omega S^{4n-1}
\end{tikzcd}$$

in which the bottom line is James' EHP fibration sequence [7]. We can factor $j \wedge j$ as $(1_{S^{2n-1}} \wedge j) \circ (j \wedge 1_{C(n)})$. The first of these is null homotopic because $n > 1$ and $S^2 j \simeq 0$ and so $j \wedge j \simeq 0$. Thus $\Omega S j$ lifts to a map $\lambda : \Omega S C(n) \to S^{2n-1}$ such that $E \circ \lambda \simeq \Omega S j$. Now $E^2 \circ \lambda = \Omega E \circ E \circ \lambda \simeq \Omega E \circ \Omega S j = \Omega(E \circ S j)$ and $E \circ S j$ is the adjoint of the null homotopic map $E^2 \circ j$. So $E^2 \circ \lambda \simeq 0$ and thus λ lifts to a map $r : \Omega S C(n) \to C(n)$ such that $j \circ r \simeq \lambda$. Theorem 1.1 follows immediately from the following theorem and Theorem 3.1.

THEOREM 2.1. *r induces an isomorphism on π_{4n-3}.*

PROOF. Let $\alpha : C(n) \to \Omega S C(n)$ denote the canonical map. From James' EHP fibration sequence, we obtain a principal fibration $\Omega^2 S^{4n-1} \to S^{2n-1} \xrightarrow{E} \Omega S^{2n}$. Let $\phi : \Omega^2 S^{4n-1} \times S^{2n-1} \to S^{2n-1}$ denote the action of the fibre on the total space in this principal fibration. Because $E \circ \lambda \circ \alpha \simeq E \circ j$, there exists $d : C(n) \to \Omega^2 S^{4n-1}$ such that $\phi \circ (d, \lambda \circ \alpha) \simeq j$. However $\pi_{4n-3}(C(n))) \cong Z/2Z$ and $\pi_{4n-3}(\Omega^2 S^{4n-1}) \cong Z_{(2)}$ and so d induces zero on π_{4n-3}. Thus $\lambda_{\#}\alpha_{\#} = j_{\#}$ on π_{4n-3}.

CASE 1. *$n \neq 2$ or 4.*

In this case the map $j_{\#} : \pi_{4n-3}(C(n)) \to \pi_{4n-3}(S^{2n-1})$ is a monomorphism (the nonzero element going to $[\iota_{2n-1}, \iota_{2n-1}]$) and so from the fact that $j_\# r_\# \alpha_\# = \lambda_\# \alpha_\# = j_\#$ we can conclude that $r_\# \alpha_\# = I$ and so r induces an isomorphism on π_{4n-3}.

CASE 2. *$n = 2$ or 4.*

In these Hopf-invariant 1 cases, $j_\# = 0$ on π_{4n-3} so we must argue differently. Since $n = 2$ or 4, $E : S^{2n-1} \to \Omega S^{2n}$ has a retraction. Thus the fact that $E \circ j \circ r \circ \alpha \simeq E \circ \lambda \circ \alpha \simeq E \circ j$ implies that $j \circ r \circ \alpha \simeq j$.

CASE 2a. *$n = 4$.*

From Toda's computations [10], the portion of the long exact homotopy sequence

$$\pi_{18}(S^7) \to \pi_{18}(\Omega^2 S^9) \to \pi_{17}(C(4)) \to \pi_{17}(S^4) \to \pi_{17}(\Omega^2 S^9)$$

is

$$Z_8 \oplus Z_2 \xrightarrow{\cong} Z_8 \oplus Z_2 \to \pi_{17}(C(4)) \to Z_8 \oplus Z_2 \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} Z_8 \oplus Z_2.$$

Thus $\pi_{17}(C(4)) \cong Z_2$ and $j_\# : \pi_{17}(C(4)) \to \pi_{17}(S^7)$ is a monomorphism. So $j \circ r \circ \alpha = j$ implies that $r_\# \circ \alpha_\#$ is the identity $\pi_{17}(C(4))$. But then $r_\#$ is an isomorphism on π_{13} since if $r_\#$ is zero on π_{13} we get a contradiction by letting $h = r \circ \alpha$ in Corollary 3.4.

CASE 2b. *$n = 2$.*

From Toda's computations, we have that

$$\pi_8(S^3) \to \pi_8(\Omega^2 S^5) \to \pi_7(C(2)) \to \pi_7(S^3) \to \pi_7(\Omega^2 S^5)$$

is

$$Z_2 \xrightarrow{0} Z_2 \to \pi_7(C(2)) \to Z_2 \xrightarrow{0} Z_2.$$

Thus $\pi_7(C(2))$ has 4 elements and $j_\# : \pi_7(C(2)) \to \pi_7(S^3)$ is onto. The inclusion $i : S^5 \cup_2 e^6 \to C(2)$ induces an isomorphism in degree 7 for connectivity reasons so we see that $\pi_7(C(2)) \cong \pi_7(S^5 \cup_2 e^6) \cong Z_4$. Since $j \circ r \circ \alpha = j$, we get $r_\# \circ \alpha_\# = uI$ where $u = 1$ or 3 and I denotes the identity. But then $r_\# \alpha_\# - I$ cannot be an isomorphism so we must have $r_\#$ nonzero on π_5 since $r_\# = 0$ on π_5 would again contradict Corollary 3.4 with $h = r \circ \alpha$. □

3. **Atomicity of $C(n)$.** Throughout this section, $n > 1$, all spaces and maps are assumed to have been localized at 2, and all homology is understood to be with $Z/2Z$ coefficients.

THEOREM 3.1. *Let $f : C(n) \to C(n)$ be such that f induces an isomorphism on $H_{4n-3}(C(n))$. Then f is a homotopy equivalence.*

PROOF. From the defining fibration of $C(n)$, we get an induced map $\theta : \Omega^3 S^{2n+1} \to C(n)$ which induces a surjection or homology. Let x denote the nonzero element of $H_{2n-2}(\Omega^3 S^{2n+1})$. Then

$$H_*(\Omega^3 S^{2n+1}) \cong Z/2Z\Big[\big\{\underbrace{Q_1 \dots Q_1}_{j}\ \underbrace{Q_2 \dots Q_2}_{k} x\big\}_{\substack{j \ge 0\\ k\ge 0}}\Big]$$

as a Hopf algebra, with the generators primitive, where Q_i denotes a Dyer-Lashof operation written in lower notation. Let $a(j, k) = \theta_*(\underbrace{Q_1 \dots Q_1}_{j}\ \underbrace{Q_2 \dots Q_2}_{k} x)$. Then $H_*(C(n)) \cong Z/2Z[\{a_{jk}\}_{(j,k)\neq(0,0)}]$ as a coalgebra with the generators primitive. From the Nishida relations the actions of Sq^1_* and Sq^2_* are given by

$$Sq^1_* a(j,k) = \left\{\begin{array}{cl} a(j-1,k)^2 & j \ge 2 \\ 0 & j = 1 \\ a(1,k-1) & j = 0,\ k \ge 1 \end{array}\right\},$$

$$Sq^2_* a(j,k) = \left\{\begin{array}{cl} 0 & j \ge 2 \\ a(2,k-1) & j = 1,\ k \ge 1 \\ 0 & j = 1,\ k = 0 \\ a(0,k-1)^2 & j = 0,\ k \ge 2 \\ 0 & j = 0,\ k = 1 \end{array}\right\}.$$

If f fails to induce an isomorphism on homotopy, let N be the least dimension in which it fails to do so. Let F be the homotopy-fibre of the canonical map from $C(n)$ to the infinite mapping-telescope $\lim\limits_{\overrightarrow{f}} C(n)$. Proceeding as in [6] or [8], we see that F is $(N-1)$-connected and that there is a map $S^N \cup_{2^r} e^{N+1} \to F$ for some r such that the composite

$$H_*(S^n \cup_{2^r} e^{N+1}) \to H_*(F) \to H_*(C(n))$$

is a monomorphism in degrees $N, N+1$. Thus in $H_*(C(n))$, we have a pair of non-zero homology classes y and z in consecutive dimensions N, $N+1$ such that

(1) y and z are primitive,

(2) $\beta^{(r)} z = y$ for some r,

(3) $y \in \ker Sq^{2^t}_*$ for all t, and $z \in \ker Sq^{2^t}_*$ for all $t \ge 1$.

By inspection of the action of Sq^1_* and Sq^2_* the only possibilities are $y = a(1,0) = \theta_*(Q_1 x)$, $z = a(0,1) = \theta_*(Q_2 x)$ or else $y = a(1,0)^2 = \theta_*(Q_0 Q_1 x)$, $z = a(2,0) = \theta_*(Q_1 Q_1 x)$. By hypothesis f induces an isomorphism in degree $4n-3$ so that we cannot have $y = \theta_*(Q_1 x)$. Therefore to prove the theorem it suffices to prove the following Lemma which shows that degree $8n-6$ is also not the least nonvanishing degree.

LEMMA 3.2. *Under the hypothesis of Theorem 3.1*, $f_*(\theta_*(Q_0 Q_1 x)) = \theta_*(Q_0 Q_1 x)$.

PROOF. For ease of notation, we will suppress θ_* and $\Omega\theta_*$ throughout this proof. Consider $\Omega f\colon \Omega C(n) \to \Omega C(n)$ and let $w \in H_{2n-3}(\Omega^4 S^{2n-1})$ denote the nonzero element. Since $f_*(Q_1 x) = Q_1 x$, commutativity with Sq^1_* forces $f_*(Q_2 x) = Q_2 x$. Thus we must have $\Omega f_*(Q_3 w) = Q_3 w$. Since Ωf is an H-map, this forces $(\Omega f_*)(Q_0 Q_3 w)$

$= Q_0Q_3w$ and commutativity with Sq^1_* now forces $\Omega f_*(Q_1Q_3w) = Q_1Q_3w$. Applying Sq^2_* now gives $(\Omega f)_*(Q_1Q_2w) = Q_1Q_2w$ and so applying the homology suspension gives $f_*(Q_0Q_1x) = Q_0Q_1x$ as desired. □

This concludes the proof that $C(n)$ is atomic. In Section 2, we also made use of the following extenison of this result to handle the cases $n = 2$ and $n = 4$.

THEOREM 3.3. *Let* $g : \Omega C(n) \to \Omega C(n)$ *be such that* g *induces an isomorphism on* $H_{4n-4}(C(n))$. *Then* g *is a homotopy equivalence.*

PROOF. The proof is similar to that of Theorem 3.1 but is slightly more tedious. It will be left as an exercise. □

COROLLARY 3.4. *Let* $h : C(n) \to C(n)$ *be such that* h *induces zero on* $H_{4n-3}C(n))$. *Then* $h_\# - 1$ *is an isomorphism on* π_q *for all* q.

PROOF. Apply Theorem 3.3 with $g = (\Omega h)(1_{\Omega C(n)})^{-1}$ in $[\Omega C(n), \Omega C(n)]$. □

REMARK 1. To obtain as much of Corollary 3.4 as is needed in Section 2, one need only demonstrate that g, satisfying the hypotheses of Theorem 3.3, induces an isomorphism on homology through $H_{18}(\ )$ when $n = 4$ and through $H_8(\ )$ when $n = 2$. The $n = 4$ case is a complete triviality since after $H_{13}(\Omega C(4))$ the next lowest nonvanishing homology group is $H_{24}(\Omega C(4))$ and the $n = 2$ case is not that much harder.

REMARK 2. Theorem 3.1 actually follows from Theorem 3.3 by letting $g = \Omega f$.

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